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Master Thesis

二維結的平滑化

Smoothings of Knot Diagrams for 2-dimensional Knots

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摘要

在 Khovanov's theory 中, 利用結的平滑化, 得到了一個 chain complex, 更進一步的可以得到一個結的不變量, 稱它爲 Khovanov's homology。

但在 Bar-Natan 教授的一篇文章中, 曾用另一個方式重新解釋這個 chain complex, 他先不將每一個平滑化的圖, 看作向量空間, 反而用 cobordism 作爲它的 differential。這是一個更抽象的 chain complex, 但很特別。這似乎是從一個更原始的角度來看此種 chain complex。

本文描述了我們將這個方法推廣到曲面嵌入四維空間 (2-knots) 的一些結果及遇到的困難, 其中也包括如何平滑化曲面圖和一些在 Roseman moves 間的 chain homotopy equivalence。

Abstract

The Khovanov's homology is the most powerful knot invariant up to now. In [1], Prof. Bar-Natan gives a new idea to interpret the Khovanov's homology. We wonder whether we can mimic his method and apply to the 2-dimensional knots. In this article, we present some results we found, and some difficulties we encountered.

Contents

1 Introduction

In Khovanov's theory, he introduces a homological invariant of knots by smoothing a knot diagram of a given knot. Roughly speaking, it is obtained by first smoothing a knot diagram, and then putting all smoothings in order. In the next step, we assign each circle a graded vector space, and define a proper homomorphism between two smoothings as its differential. The homology group of this chain complex turns out to be a knot invariant and we call it Khovanov's homology of this knot. Bar-Natan gives a clear introduction of this homological invariant in [2].

In the paper [1] of Prof. Bar-Natan, he found a new interpretation of these chain complexes. He uses the cobordisms to be the differentials and directly considers the direct sum of the smoothings as the chain groups. It is a very interesting and special idea because all the maps are not "real maps" as functions but are the surfaces with boundary. Moreover, it seems to me that he interprets the Khovanov's homology from a more foundamental point of view.

We will review Bar-Natan's interpretation for Khovanov's homology in the section 2. Many definitions used to define the Khovanov's bracket, as well as the idea of the proof for Khovanov's bracket to be a knot invariant will be contained in section 2.

In section 3, we will review some basic definitions about 2-dimensional knots. The character of a 2-knot diagram and the definition of Roseman moves which is an analogy of Reidemeister moves will be contained in this section. For more details about 2-knots, we refer to Carter's book [3].

In the last two sections, we want to mimic Bar-Natan's method in [1] and apply it to the 2-dimensional knot diagrams. We will present some basic definitions which are an analogy of those for 1-dimensional knots in section 2, and also define the category Cob^4 whose morphisms are 3-dimensional cobordisms. In order to understand these 3-dimensional cobordisms, we use the movie method which will be described in section 4.3.

Finally, some homotopy equivalences between Roseman moves and some problems we encountered will be described in the last section.

2 1-dimensional case

Most parts in this section can be found in [1] and we just review his construction.

2.1 Review

We now begin to review some basic definitions which Prof. Bar-Natan used to interpret the Khovanov's chain complex.

Definition 2.1. A tangle is an image of an embedding which embeds some arcs or S^1 into B^3 with endpoints of these arcs lying in the boundary of B^3. A tangle diagram is the image of a generic projection from B^3 onto B^2 which will send the tangle into B^2 with that each intersection is transversal and the ends of these arcs lie in the boundary of B^2.

Definition 2.2. A labeled oriented tangle diagram is a tangle diagram which we give them an orientation and number all its crossings.

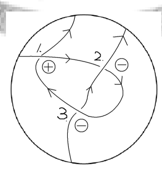

Figure 1: Labeled oriented tangle

Definition 2.3. Given a knot diagram, we also mark the sign of its crossings – (+) for overcrossings and (−) for undercrossings.

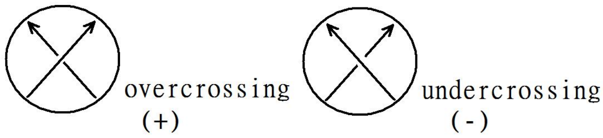

Figure 2:

Definition 2.4. The 0-smoothing and 1-smoothing: a crossing is an interchange involving two highways.
The 0-smoothing is when you enter on the lower level and turn right at the crossing.
The 1-smoothing is when you enter on the upper level and turn right at the crossing.

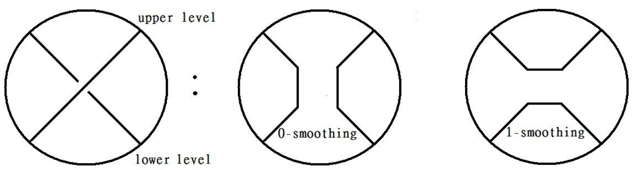

Figure 3:

Now, for a given n crossing knot we can get a n-dimensional cube with vertices $\{0,1\}^n$, projected by the "shifted height" to the interger points on $[-n_-, n_+]$. And the edges of the cube are marked in the natural manner by n-letter strings of 0's, 1's and precisely one $\star$. The $\star$ denote the coordinate which changes from 0 to 1 along a given edge.

Vertices: Each vertex of the cubes carries a smoothing of K (the given labeled oriented knot diagram)– a planar diagram obtained by replacing every crossing in the given diagram of K with either a "0-smoothing" or a "1-smoothing".

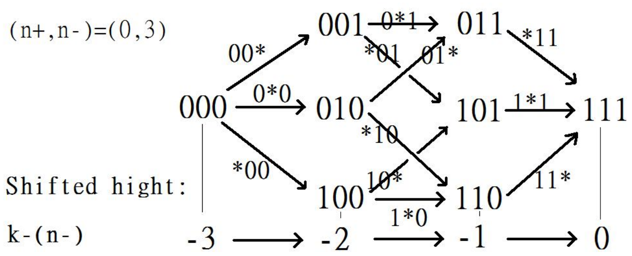

Figure 4: n-cube

Shifted height: If a vertex ζ is labeled by a sequence (ζ_i) in the alphabet $\{0, 1\}$, then its height $k = \sum \zeta_i$ and the "shifted height" is $k - n_-$.

Edges: Each edge of the cube is labeled by a cobordism between the smoothing on the tail of that edge and the smoothing on its head– an oriented two dimensional surfaces embedded in $R^2 \times [0, 1]$ whose boundary lies entirely in $R^2 \times$ 0, 1 and whose "top" boundary is the "tail" smoothing and whose "bottom" boundary is the "head" smoothing. Specifically, to get the cobordism for an edge $\xi \in \{0, 1, \star\}^n$ for which $\xi_j = \star$ we remove a disk neighborhood of the crossing j from the smoothing $\xi(0) \equiv \xi \mid_{\star=0}$ of K, cross with [0,1], and the empty cylindrical slot around the missing crossing with a saddle cobordism (Figure 5).

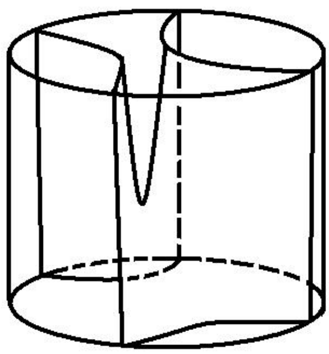

Figure 5: Saddle cobordism

Sign: We give a sign on each edge. If an edge ξ is labeled by a sequence (ξ_i) in the alphabet $\{0, 1, \star\}$ and if $\xi_j = \star$, then the sign on the edge ξ is $(-1)^{\sum_{i<j} \xi_j}$.

Now, we will define a new category which plays an important role in this new intepretation for Khovanov's homology.

Definition 2.5. The $Cob^3(B)$ (or $Cob^3(\emptyset)$) is a category whose objects are smoothings with boundary B (or $\emptyset$) and whose morphisms are the 2-dimensional cobordisms between such smoothings, regarded up to boundary-preserving isotopies. The composition of morphisms is given by placing one cobordism atop the other.
(We will use the notation Cob^3 as a generic reference either to $Cob^3(B)$ or to $Cob^3(\emptyset)$.) There is a picture describing the morphism and the objects in Cob^3.

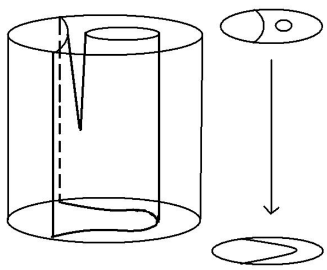

Figure 6: A morphism between two smoothings

The $Cob^3_{/l}$ is a qoutient of Cob^3 by adding some relations in the morphisms of Cob^3. They are the following three relations:

- Sphere$\sim$0.
- Torus$\sim$2.
- 4Tu-relation can be described by following picture.

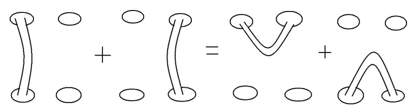

Figure 7: 4Tu-relation

Remark 2.6. In fact, we can think of the $Cob^3_{/l}$ as an additive category by extending every set of morphisms $Mor(\mathcal{O}, \mathcal{O}')$ to also allow of fromal $\mathbb{Z}$-linear combinations of "original" morphisms and by extending the composition maps in the nature bilinear manner.

Note that an additive category is a category in which the sets of morphisms are Abelian groups and the composition maps are bilinear in the obvious sense.

Definition 2.7. Given an additive category $\mathcal{C}$, we can define another additive category $Mat(\mathcal{C})$:

- The objects of $Mat(\mathcal{C})$ are formal sums(possibly empty)$\bigoplus_{i=1}^{n} \mathcal{O}_i$ of objects $\mathcal{O}_i$ of $\mathcal{C}$.
- If $\mathcal{O} = \bigoplus_{i=1}^{n} \mathcal{O}_i$ and $\mathcal{O}' = \bigoplus_{i=1}^{n} \mathcal{O}'_i$, then a morphism $F : \mathcal{O}' \to \mathcal{O}$ in $Mat(\mathcal{C})$ will be an $m \times n$ matrix $F = (F_{ij})$ of morphism $F_{ij} : \mathcal{O}'_j \to \mathcal{O}_i$.

- Compositions of morphisms in $Mat(\mathcal{C})$ are defined by a rule modeled on matrix multiplication, but with compositions in $\mathcal{C}$ replacing the multiplication of scalars,$((F_{ij}) \circ (G_{jk})) \equiv \sum_j F_{ij} \circ G_{jk}$.

Note that It is convenient to represent objects of $Mat((C))$ by column vectors. The following picture is an example of $Mat(Cob^3_{/l})$

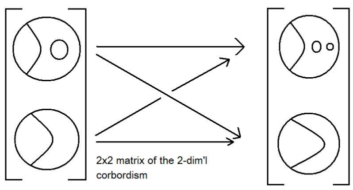

Figure 8: $Mat(Cob^3_{/l})$

Definition 2.8. Given an additive category $\mathcal{C}$, we define $Kom(\mathcal{C})$ to be the category of complexes over $\mathcal{C}$, whose objects are chains of finite length $... \to \Omega^{r-1} \to^{d^{r-1}} \Omega^r \to^{d^r} \Omega^{r+1} \to ...$ for which the composition $d^r \circ d^{r-1}$ is 0 for all r, and whose morphism $F = (F^r)$ satisfy the commutativity $(F^{r+1} \circ d^r = d^r \circ F^r)$. The composition is defined via $(F \circ G)^r \equiv F^r \circ G^r$.

Note that we also can define what is "Two morphisms in $Kom(\mathcal{C})$ are homotopic.", "A chain map is a homotopy equivalence.", and "Two chain complexes are equivalent.".

Now, we can define the Khovanov's bracket [ ] which is a function from the tangle diagram to the category of $Kom(Mat(Cob^3_{/l}))$. For a given tangle diagram T, we can interpret the n-cube (n-dimensional cube of smoothings) as a chain complex denoted $[T]$ by thinking of all smoothings as objects in $Cob^3_{/l}$ and of all cobordisms as the morphism in $Cob^3_{/l}$. We set the r's chian space $[T]^k$ of the complex $[T]$ to be the direct sum of those smoothings whose shifted height is k (which is an object in $Mat(Cob^3_{/l})$) and to sum those cobordisms from the smoothings at height-$(k-1)$ to those at height-k in the cube to get the differential (which is a morphism in $Mat(Cob^3_{/l})$). ($i.e.$, $[T] =\longrightarrow [T]^{-n_-} \to [T]^{-n_-+1} \to ..., \to [T]^{n_+-1} \to [T]^{n_+}$)

This is an example of $[T]$.

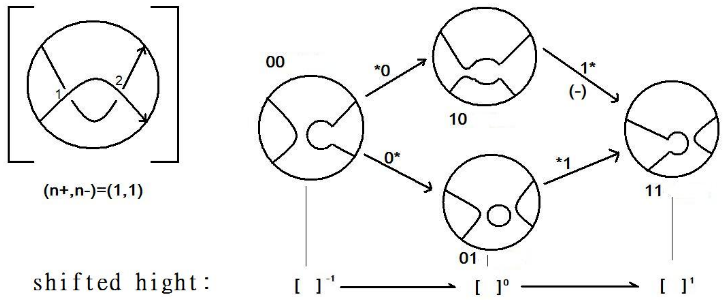

Figure 9: Khovanov's bracket of a tangle diagram

It is convenient to short Kob for $Kob(Mat(Cob^3_{/l}))$ in the next subsection.

2.2 Invariance

Before we begin this section, we must recall that two tangle diagrams are equivalent iff one can be obtained by another via finite Reidemeister moves. The main theorem which I will state in this section tells us that the Khovanov's bracket can be viewed as a tangle invariant. It means that we must show that the Khovanov's brackets of two equivalent tangle diagrams are two homotopy equivalent chain complexes.
Notice that the homotopy equivalence here will be some 2-dim'l cobordisms, for we are working in the category Kob $(Kob(Mat(Cob^3_{/l})))$.

In this section, we first state the main theorem, and then write down four lemmas to make the scheme of our proof clearer. We won't carry out all the proof of them, so if you want to get more details about the proof of these lemmas, you can refer to the Bar-Natan's paper [1]. I just rewrite his idea and emphasize what will be used in the following sections.

Notation 2.9. We list some useful notations below:

- $\mathcal{T}^0(k)$ denote the collection of all k-ended unoriented tangle diagram.

- $\mathcal{T}^0(s)$ denote the collection of $\mid s \mid$-ended oriented tangle diagram whose incoming/outgoing strands is specified by s.
  (s is a sting of in($\uparrow$) and out ($\downarrow$) symbols with a total lengh of $\mid s \mid$).

- $\mathcal{T}(k)$ denote the quotient of $\mathcal{T}^0(k)$ by three Reidemeister moves.

- $\mathcal{T}(s)$ denote the quotient of $\mathcal{T}^0(s)$ by three Reidemeister moves.

- $Kob(k) \equiv Kom(Mat(Cob^3_l(B_k)))$($B_k$ is k points on the boundary.)

- $Kob_{/h} \equiv Kob/$(homotopy equivalent chain complexes)

Theorem 2.10. The Khovanov's bracket [ ] can be thought of as a function from $\mathcal{T}$ to $Kob_{/h}$. (i.e.It means if T_1 and T_2 are two equivalent tangle diagrams, than $[T_1]$ and $[T_2]$ are homotopy equivalent.)

Our main strategy is first to construct homotopy equivalences between three Reidemeister moves and then to reduce the global case to local case which is just one of three Reidemeister moves.
In order to reduce the global case to the local case, he introduces the "planar algebra" which will be defined later.

Lemma 2.11. The Khovanov's bracket [ ] is invariance under three Reidemeister moves.

For example:(we want to show these two chain complexes are homotopy equivalent)

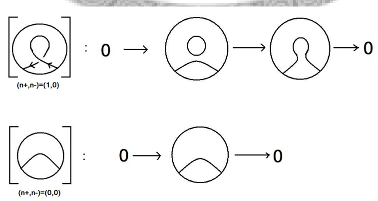

Figure 10: First type

Proof. We only verify the type1 case. The proof of the other cases can be found in [1]. In the proof for type1 case, we can find how the relations: **Torus∼2** and $4Tu$**-relation** work. In fact, they play an very important role in constructing homotopy equivalence.

We will construct the maps A, B, and H in the figure11 and then check that it is a homotopy equivalence between them.

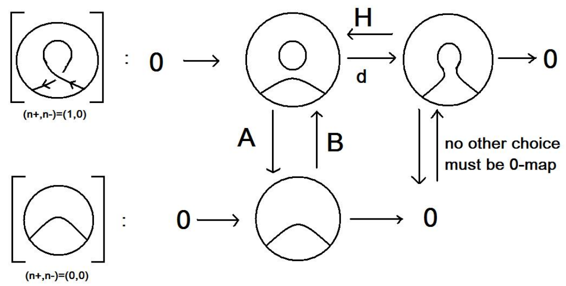

Figure 11: Search homotopy equivalence

The maps: A, $B = B_1 - B_2$ and H are:
Notice that the top of each can corresponds to the smoothing on the tail of

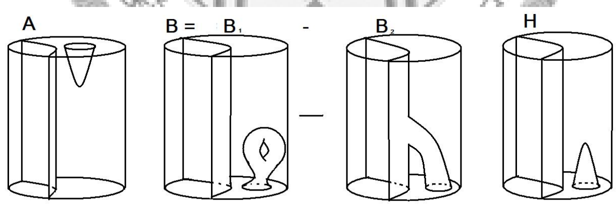

Figure 12: Maps: A, B and H

some arrow and the coefficient of H is negative one.

First, we check that whether they are chain maps.

$d \circ A = 0 \circ d$ is OK.

$d \circ (B_1 - B_2) = (d \circ B_1) - (d \circ B_2) = 0 = 0 \circ d$.

The picture of the equality is like:

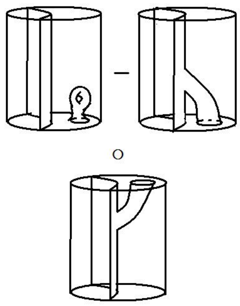

Figure 13: Both of them are the same manifold

Secondly, we check that they are a homotopy equivalence:

$A \circ (B_1 - B_2) = A \circ B_1 - A \circ B_2$.

Since we have the relation $Torus \sim 2$

and $A \circ B_2 = Id$,

so we have $A \circ (B_1 - B_2) = 2 \cdot Id - Id = Id$.

For the case of $B \circ A$ we use $4Tu$-relation on this holed 2-corbordism.

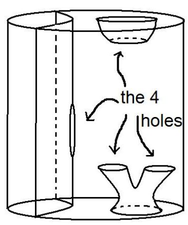

Figure 14: How $4Tu$-relation works here

Then we obtain the equality:

$B_1 \circ A - B_2 \circ A - Id = H \circ d$.

Hence we get a homotopy equivalence for first Reidemeister move. $\square$

Note: We will use the relations: $Sphere = 2$ and $4Tu$-relation to construct the homotopy equivalence for second Reidemeister move. And the homotopy equivalence for third Reidemeister move can be obtain from the second Reidemeister move directly by some technique.

Definition 2.12. A d-input planar arc diagram D is a big "output" disk with d "input" discs removed, along with a collection of disjoint embedded oriented arcs that are either closed or begin and end on the boundary. The input discs are numbered 1 through d, and there is a base point(*) marked on each input disc and the output disc.
(Similarly, we can define the unoriented planar arc diagram by forgetting the orientation of arcs.)

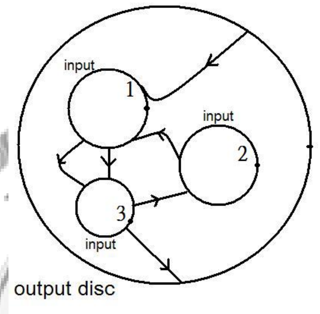

Figure 15: Planar arc diagram

Definition 2.13. A planar algebra is a collection of sets $(\mathcal{P}(k))$(or$\mathcal{P}(s)$) which has these properties:
(1)We can define an operation $D : \mathcal{P}(k_1) \times \mathcal{P}(k_2) \times ... \times \mathcal{P}(k_d) \to \mathcal{P}(k)$ for each d-input planar arc diagram D.
(2)The operation defined for the identity diagram must be identity.

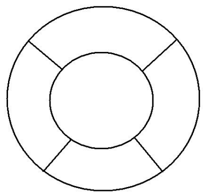

Figure 16: Identity planar arc diagram

(3)The associative law: If D_i is the result of placing D' into the i-th hole of D, then as operations,$D_i = D \circ (I \times ... \times D') \times ... \times I$.

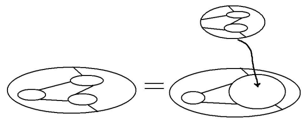

Figure 17: Associative law

For example the tangle diagram space $\mathcal{T}^0$ is a planar algebra.

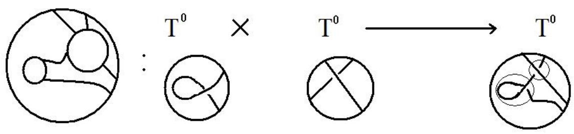

Figure 18: Tangle diagram space

I list some planar algebras we will use here:

1. Oriented planar algebra

   - $\mathcal{T}^0(s)$
   - $\mathcal{T}(s)$

2. Unriented planar algebra

   - $Obj(Cob^3_{/l})$
   - $Mor(Cob^3_{/l})$
   - $Obj(Mat(Cob^3_{/l}))$
   - $Mor(Mat(Cob^3_{/l}))$

We note that every unoriented planar algebra can be regarded as an oriented one by setting $\mathcal{P}(s) \equiv \mathcal{P}(| s |)$.

Lemma 2.14. The collection of the objects in $Kob(s)$ is a planar algebra.

Proof. Given a planar diagram D which has n-input discs and an output disc.
We can define a function D:$\times_n Kob \rightarrow Kob$ by the following way: (like tensor). Let Ω_i =(Ω_i^r) denote the n chain complexes for $i = 1, ..., n$. Then we define:

$$
\begin{aligned}
&\Omega \equiv D(\Omega_1, \Omega_2, ..., \Omega_n) \\
&\Omega^r \equiv \bigoplus_{r=r_1+...+r_n} D(\Omega_1^{r_1}, \Omega_2^{r_2}, ..., \Omega_n^{r_n}) \\
&d\, |_{D(\Omega_1^{r_1},\Omega_2^{r_2},...,\Omega_n^{r_n})} \equiv \sum_{i=1}^{i=n} (-1)^{\Sigma_{j<i} r_j} D(I_{\Omega_1^{r_1}}, ..., d_i, ..., I_{\Omega_n^{r_n}})
\end{aligned}
$$

□

Lemma 2.15. We list some properties of such planar algebra.:

- $D(I, ..., F_i, ..., I)$ induce a function,$D(\Omega_1, ..., \Omega_{ia}, ..., \Omega_n) \rightarrow D(\Omega_1, ..., \Omega_{ib}, ..., \Omega_n)$ as F_i:$\Omega_{ia} \rightarrow \Omega_{ib}$
- $D(I, ..., F, ..., I) \circ D(I, ..., G, ..., I)$=$D(I, ..., F \circ G, ..., I)$
- If $F_1 + ... + F_n = G_1 + ... + G_m$ by some of the relations in $Cob^3_{/l}$ then $D(I, ..., F_1, ..., I)+, ..., +D(I, ..., F_n, ..., I) = D(I, ..., G_1, ..., I)+, ..., +D(I, ..., G_m, ..., I)$

It can be shown by observing how the cobordism be put into a planar arc diagram.

Lemma 2.16. The Khovanov's bracket [ ] has such commutative law:
$[D(T_1, ..., T_n)] = D([T_1], ..., [T_n])$

Proof. We can consider the case: All $T_1, ..., T_n$ are single crossings. This is true since it $(D([T_1], ..., [T_n]))$ just is the definition of Khovanov's bracket. So by associative law of planar algebra, It is still true for arbitrary tangles$(T_1, ..., T_n)$.

□

Now, we can prove the main theorem:
Recall that we want to show that the Khovanov's bracket is a function from tangle space to $Kob_{/h}$ (Kob/homoyopy equivalence).

Proof. Suppose T_1 and T_2 are equivalent tangles in $\mathcal{T}^0$. It means they differ from each other by finite Reidemeister moves. So we can assume that their difference is just one of these three Reidemeister moves. We can circle all its crossing so that $T_1 = D(T, T^1, ..., T^n)$, $T_2 = D(T', T^1, ..., T^n)$ and T and T' is one of the three Reidemeister moves.
Since we have lemma 4,
so $[T_1] = [D(T, T^1, ..., T^n)] = D([T], [T^1], ..., [T^n])$.
By lemma 1, we have homotopy equivalence F between $[T]$ and $[T\prime]$.
By lemma 3, we can verify that $D(F, I, ..., I)$ is still a homotopy equivalence of $D([T], [T^1], ..., [T^n])$ and $D([T\prime], [T^1], ..., [T^n])$.
But we have $D([T\prime], [T^1], ..., [T^n]) = [T_2]$,
so $[T_1] \sim [T_2]$. $\square$

The following picture shows that how we circle those crossings:

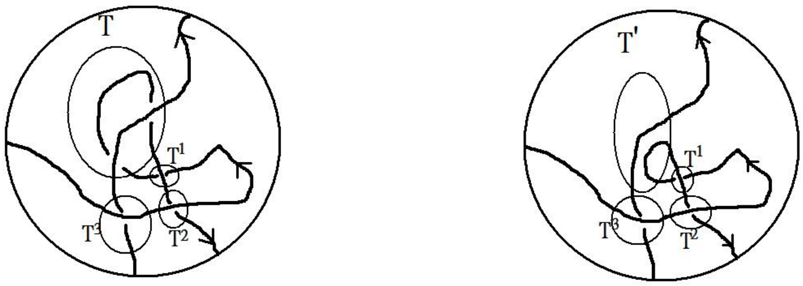

Figure 19:

Remark 2.17. If we consider the case of knot, then the smoothings of them will not have ends on the boundary. It means each smoothing is some circles in B^2. So we can assign a graded vector space to each circle in each

smoothing, and a proper linear map to each edge of the cube, then we will obtain a chain complex over the category of vector spaces. After shifting artificially the degree of those graded vector spaces, we will get Khovanov's homology from the complex. Moreover, the homology group's Eular characteristic which is a polynomial is the Jones polynomial.
More details can be found in [1] and [2].

3 Some basic definitions in 2-knots

In this section, we review some basic definitions about 2-knots. Since we usually discuss 2-knots by their diagrams, so one of the points in this section is to describe what the basic components of a 2-knot diagram are. The other important terminology is the Roseman moves which is an analogy of Reidemeister moves. Roseman moves, which has seven members, plays an important role in defining a 2-knot invariant by 2-knot diagrams.

Definition 3.1.

- A 2-knot is an embedding $K{:}F \to \mathbb{R}^4$, where F is a orientable closed surface. Sometimes we also consider the image $K(F)$ as the 2-knot.
- We say two 2-knots: k_1 and k_2 are ambient isotopic or have the same knot type, if there exsists an isotopy $H{:}\mathbb{R}^4 \times I \to \mathbb{R}^4$ such that $H(x, 0) = x$ and $H(k_1(a), 1) = k_2(a)$, where $a \in F$ and F is a closed surface.
- We say a 2-knot is unknotting, if it is ambient isotopic to the standard embedding in $\mathbb{R}^3$.

Remark 3.2. The definition of unknot is equivalent to that the image of embedding is a boudary of some handlebodies in $\mathbb{R}^4$.

Since we want to introduce what a 2-knot diagram looks like in $\mathbb{R}^3$, which is an analogy of classical knot diagram in $\mathbb{R}^2$, we first define the notion: generic surface.

Definition 3.3. A smooth map $f : K$(a closed surface)$\to \mathbb{R}^3$ is generic
if and only if any point $y \in f(K)$ has a 3-ball neighborhood $N(y)$
such that the pair $(N(y), N(y) \bigcap f(K))$ is diffeomorphic to either (B^3, The intersection of some coordinate planes)
or (B^3, The cone on a figure eight).
The following picture can tell us what they look like.

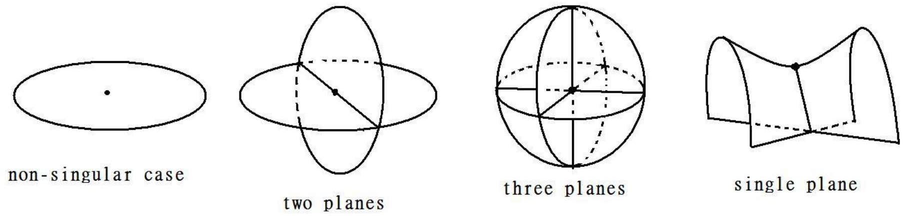

Figure 20: The local neighborhood of a generic surface

Remark 3.4.

- The term ‘generic’ has the meaning of ‘stable’.
- In fact, we can use our understanding of generic map to prove the three Reidemeister moves. It follows by changing the shape of these three types of singularity a little.

Definition 3.5. A broken surface diagram is a diagram in $\mathbb{R}^3$ which is obtained from a generic surface by:

1. Replace the intersection $f(K)\bigcap N(y)$ with some other manifolds:
   case1: If the intersection is the case of double points than we replace it by three discs. It can be obtained by removing the neighborhood of one of repeated lines.
   case2: If the intersection is the case of tripe points than we replace it by seven discs. It can be obtained by removing two lines from one of these three planes and then removing a line from one of the remainder two planes.
   case3: If the intersection is the case of branch point than we replace it by one discs. It can be obtained by removing the neighborhood of one of repeated lines, but reserve the branch point.

   Their pictures are listed below.

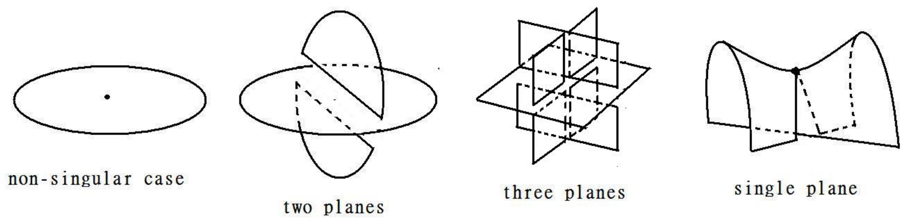

Figure 21: Four kinds of the neighborhood of broken surface diagram

2. After we have finished modifing these neighborhoods, we still require they are patched compatibly.

Notice that there is a generic surface cannot be converted into the broken surface diagram because those neighborhoods can't have a removing way to be patched compatibly.

Now, we can present some basic theorems in the 2-knot theory, but we will not prove them in detail.

Theorem 3.6. For any 2-knot, $f : K \to \mathbb{R}^4$, we can find a vector $v \in \mathbb{R}^4$ and the projection π_v, which projects $\mathbb{R}^4$ onto the plane whose normal vector is v, such that the composition $\pi_v \circ f$ is generic (or generic surface) and the generic surface has a natural way to be converted into the broken surface diagram.

Proof. We just indicate how to convert this generic surface which is the image of $\pi_v \circ f$ into a broken surface diagram: Since we have the direction v, so we can define the height of this 2-knot. We remove a neighborhood of the curve which is at the higher position among those curves whose image of $\pi_v \circ f$ is the same. Notice that we do this on the whole 2-knot in $\mathbb{R}^4$. We can find the projection of the modified 2-manifold is a broken surface diagram for all of those neighborhoods of its crossings are patched compatibly automatically.

□

For more details and some pictures about the process, we refer to the book [3] of Carter and Saito. We call the broken surface diagram of $\pi_v \circ f$ a digram of the 2-knot: f. We must also note this any broken surface diagram can be a diagram of a 2-knot.

Theorem 3.7. There is a 1-1 correspondance between {the collection of 2-knots}$/$*ambient isotopy* and {the broken surface diagram}$/$ *Roseman moves.*

The complete pictures of seven Roseman moves can be found in the book [3]. Here we only present some Roseman's moves we may use later: first type(bubble and saddle), second type(bubble and saddle).

Type1

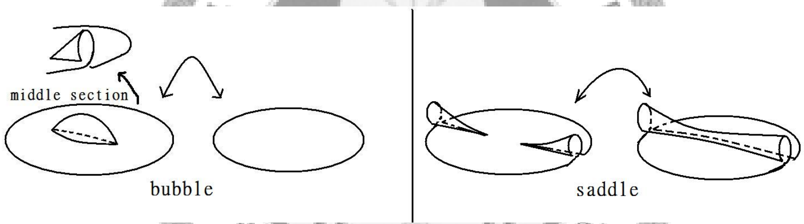

Figure 22: Some of Roseman moves

Type2

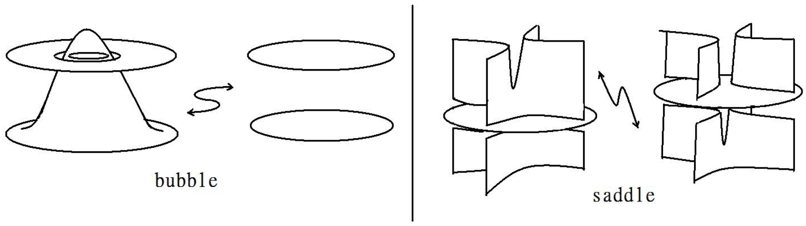

Figure 23: Some of Roseman moves

Proof. I sketch a possible process to show the theorem:
In fact, we already have a function h
from {the collection of 2-knots} / *ambient isotopy*
into {the broken surface diagram} / *Roseman moves*.
By understanding the generic map from a 3-manifold into the $\mathbb{R}^4$, we can show that the projection of any ambient isotopy is a combination of some Roseman moves. This show that h is well-defined. Since we can obtain a 2-knot from a broken surface diagram (refer to [3]), so the *onto* follows.
The *1 to 1* is due to each Roseman move can induce an isotopy between the two knots whose diagrams are different from some of Roseman moves.

□

4 Smooth 2-knot diagrams and Cob^4

In this section, we want to imitate the method which is developed for classical knots in sec.1 and by using the method we try to construct a 2-knot invariant. We divide this section into three subsections. In section 4.1, we give some definitions about 0-smoothing, 1-smoothing, postive and negative crossing and then describe the topology after smoothing those basic crossings which are defined in sec.3. But there are some problems I still can't solve or explain in these definitions.

In the last two subsections, we introduce the category Cob^4 which is an analogy of Cob^3 and we use the movie method to realize the morphisms in Cob^4.

In the following discussion, we only consider those 2-knots which are the embedding of some orientable surfaces.

4.1 Preliminary

Definition 4.1. A labelled oriented 2-knot diagram is a 2-knot diagram which satisfies the following conditions:

- It has an orientation on the surface.
- For each crossing, we have an orientation on it with some compatible requirement. The compatibility involves the Roseman moves, but I don't realize it clearly yet.
- Each crossing has been numbered.

There is an example of a labelled oriented 2-knot diagram. In fact, it is unknot.

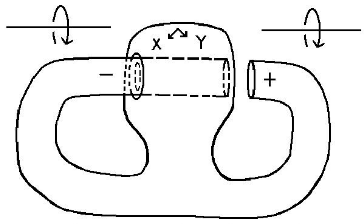

Figure 24: A labeled oriented 2-knot diagram

Definition 4.2. The positive and negative crossings in a 2-knot diagram can be defined by following way: Let the orientation of the crossing is v and the oreintation of the surface is $[e_1, e_2]$.
At a crossing, we have two planes transverse.
Let the direction e_1 of the two planes be the v and consider the orientation $[e_2(\text{under plane}), e_2(\text{upper plane}), v]$. Then compare it with the standard orientation in $\mathbb{R}^3$.
If they coincide then it is positive crossing. If not, it is negative crosssing.

Definition 4.3. 0-smoothing and 1-smoothing can be defined as following way which doesn't require giving an orientation on the surface. By the orientation on the crossing, we can project the crossing to a plane and then determine the smoothing type of the crossing as in the case of knot diagram. The following picture tells us this process.

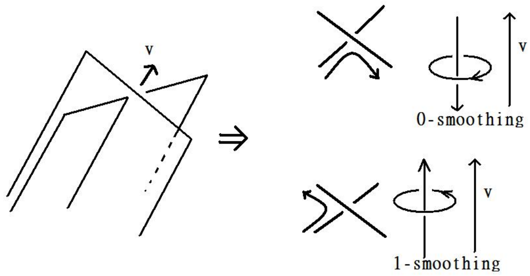

Figure 25: 0-smoothing and 1-smoothing

Remark 4.4. There are some serious problems in these definitions: For example can we have a natural way to give an orientation for those crossings. And can the orientation be compatible between each Rossman moves which will change the crossings in a 2-knot diagram.

It means I yet don't know how to find a natural oreintation for each crossing of a given 2-knot diagram. So we won't give a precise definition about the chain complex as the one in sec.2. And we directly begin to study the category Cob^4 and try to construct the homotopy equivalence between each Roseman move.

We now give an example of the chian complex of a 2-knot diagram.

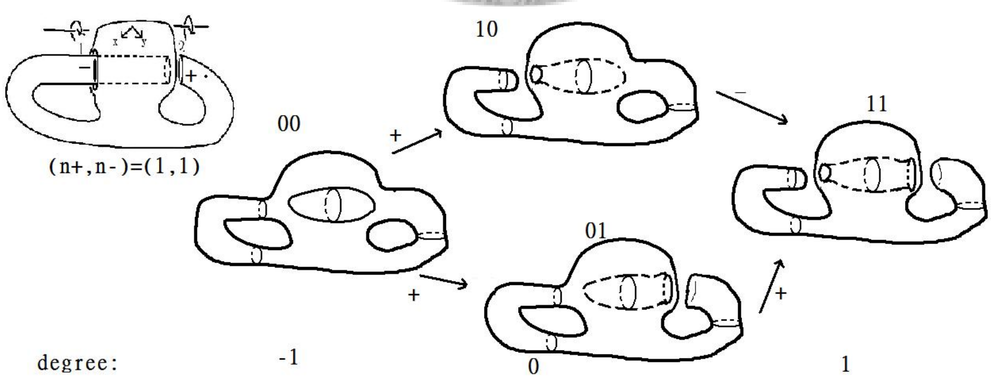

Figure 26: Chain complex for a 2-knot diagram

4.2 The topology after smoothing

In classical knot diagrams, there is only one type of crossing. But in the case of 2-knot diagrams, there are three types of crossing. In order to realize its differential, we must realize the topology after smoothing those basic crossings and the 3-cobordisms between two such smoothings. In classical case, the differential is just a disc.

Note that in the case of triple point, there are three double curves in its neighborhood. So there are not only two ways to smooth it as in other case.

Now we begin to study the topology after smoothing those basic crossings

.

- Double points:

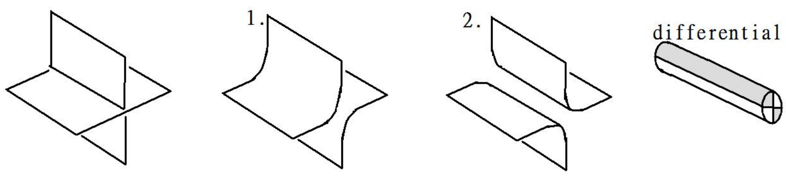

Figure 27: Double point

- Branch points:

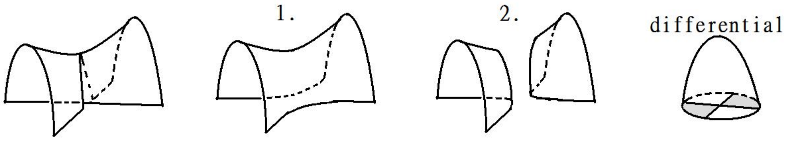

Figure 28: Branch points

- Triple points:

  It seems the case of triple point is more diffcult to realize, since it has three double curves, and each of them has two ways to be smoothed. So there are eight ways to smooth it. But the result is that there are only two topological types among them. One is a disc, and the other is three discs.

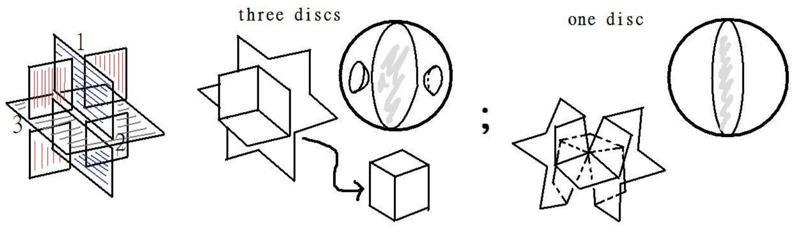

Figure 29: Triple point

Theorem 4.5. There is a process to smooth a triple point such that the eight kinds of smoothing have only two topological typies as the above picture.

Proof. First, we smooth the two double curves with fixing the triple point:

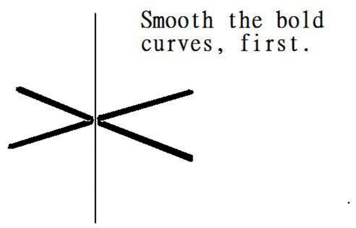

Figure 30:

Secondly, observe that no matter how we smooth these two double curves with fixing the triple point, we get the same topological space by reflecting one to the other. Hence we only have two choices to smooth the remainder double curve, and we can conclude that there are only two topological typies in these eight kinds of smoothing.

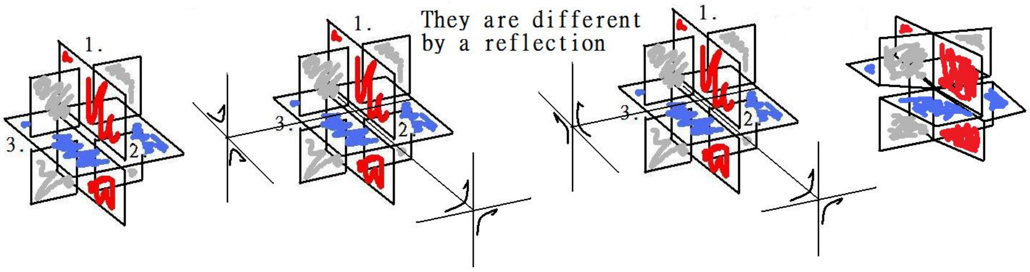

Figure 31:

Furthermore, like figure 32, The two possible ways to smooth the remainder double curve can be describe as following way: One is to smooth it along the red line, and the two cones will leave the middle disc.(figure 32) So we will get three discs.
The other is to smooth it along another direction and we will get one disc.

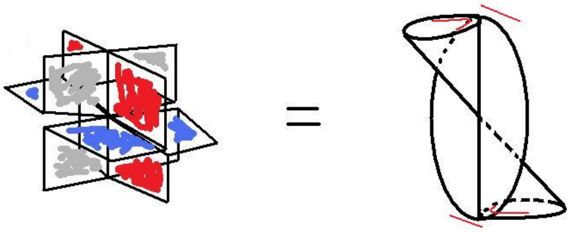

Figure 32: The topology of it

Note that the two different manifolds are those indicated before the theorem. □

Remark 4.6. The differentials in the case of triple point are more complicated than other two cases. We can observe that some of the three double curves can be possibly joined together by some arcs outside the B^3. So may have three different differentials at least: One is smooth one double curve at a time, and the others are smooth two double curves or three double curves at a time.
Here is an interesting phenomenon: Some of the 3-cobordisms between two smoothings of a triple point can't be embedded into R^3; For example smooth three double curves at a time.

4.3 The movie method

To analogize Bar-Natan's theory, we introduce the Cob^4 which is an analogy of Cob^3. Its object is the surface (with boundary) in a 3-ball. Its morphism is the 3-manifold (with boundary) embedded generically in $B^3 \times I$.
In order to describe the 3-manifold, we use the *movie method*.

To be clear we first apply this method to the 2-dimensional cobordisms which are the morphisms in Cob^3. It is an another way to describe the 2-cobordisms and it can be applied to the 3-cobordisms, too.

The main idea comes from the Morse theory [5]. It means we can use the level sets and the critical points (must be discrete) to understand a 2-dimensional cobordism.

Definition 4.7. We define these clips to be the 0-handle,1-handle and 2-handle.

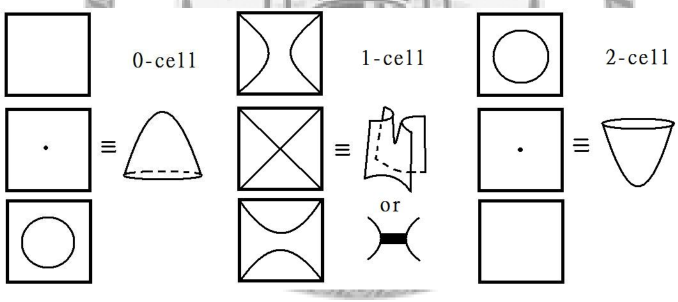

Figure 33: movie diagram

In figure 34, we describe a generic 2-cobordism as a moive.

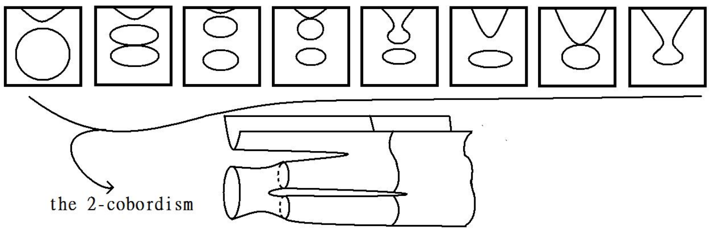

Figure 34: Movie diagram

Theorem 4.8. Any 2-cobordism embedded in $\mathbb{R}^3$ can be describe as a movie.

Proof. By Morse theory, we can find a height function which is a Morse function as we restrict it on this 2-cobordism, so we can find its level set of regular value is some 1-dimensional objects in $\mathbb{R}^2$. And the way to attach those cells when we pass though a critical point are just these three types. $\square$

Now we use the similar way to understand the 3-cobordism in $\mathbb{R}^4$. It means that we slice it into many slices in $\mathbb{R}^3$ so that we can draw. By observing the change between each slice, we can understand the 3-cobordism in $\mathbb{R}^4$.

Definition 4.9. We define following clips to be attching cells in some certain ways.

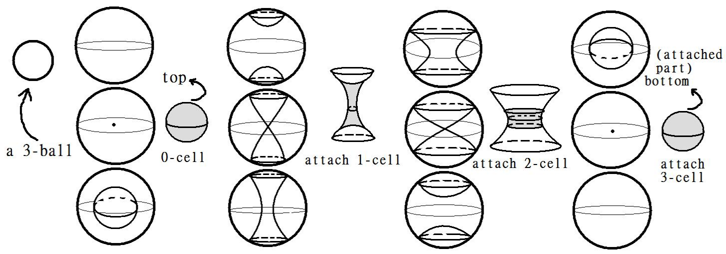

Figure 35: The basic short clips

Theorem 4.10. All 3-manifolds embedded in $\mathbb{R}^4$ can be described by a movie which is a combination of some clips described in definition 4.9.

Note that the description is not unique.

Proof. By Morse theory, we know a 3-manifold can be constructed by attaching some cells as we pass through the critical points of some Morse function. So to prove this theorem, it suffices to show that the ways to attach 1-cell and 2-cell are just these two types described in definition 3. We now rule out two other possibilities so that the attaching way are only these two typies.

1. The case in the lefthand side will imply that the non-orientable surface can be embedded in $\mathbb{R}^3$. But by the Alexander duality, it is impossible.

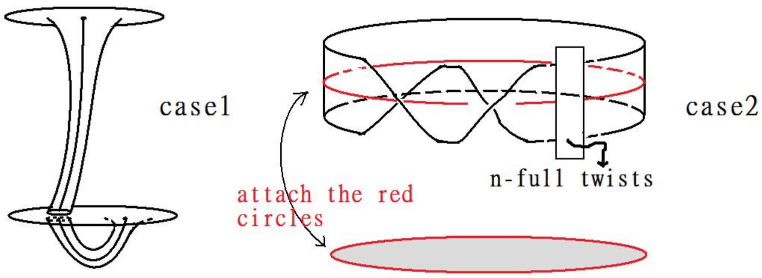

Figure 36: Two impossible cases

2. The case in the righthand side can't occur, since the boundary of the band had nonzero linking number. So one of the two circles can't be bounded by a disc without intersecting the other circle.

□

4.4 The category: Cob^4

To let the discussion in section 4 be systematic, we introduce the category Cob^4 which is an analogy of the Cob^3.

Definition 4.11. The object in Cob^4 is some surfaces embedded in a 3-ball with the intersection of the surfaces and the boundary of the 3-ball(B^3) is exactly the boundary of the surfaces.

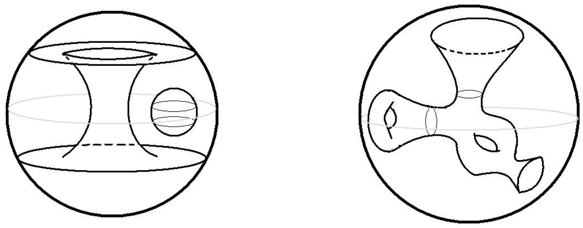

Figure 37: Objects in Cob^4

The morphism in Cob^4 is a 3-dimensional cobordism embedded in $B^3 \times I$ with the height function restricted on the 3-cobordism is a Morse function. So we can use the *Movie Method* to realize this 3-cobordism.

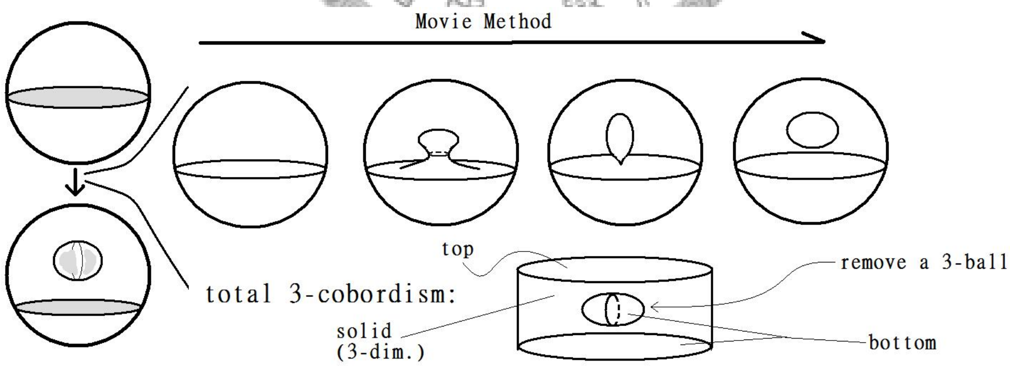

Figure 38: Morphism in Cob^4

Note that we consider the Cob^4 as an additive category. It means we can formally add two morphisms as those morphisms in Cob^3.

5 Homotopy equivalences

In this section, we present some homotopy equivalences for the first type Roseman moves and some relations we require when we construct these homotopy equivalences. But now there are still many problems we can't solve in constructing homotopy equivalence. In this section, we also summerize them.

5.1 First type Roseman move (bubble)

Here we imitate the homotopy equivalence which Bar-Natan gives in the first Reidemeister move and construct the homotopy equivalence for the first type Roseman move (bubble). We require some relations which are an analogy of the relations: $Torus \sim 2$ and $4Tu$-relation in the Cob^3. They are $S^1 \times S^2 \sim 2$ and $New4Tu$-relation:

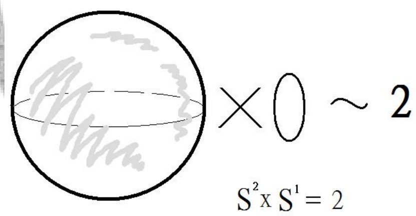

Figure 39: $S^1 \times S^2 \sim 2$

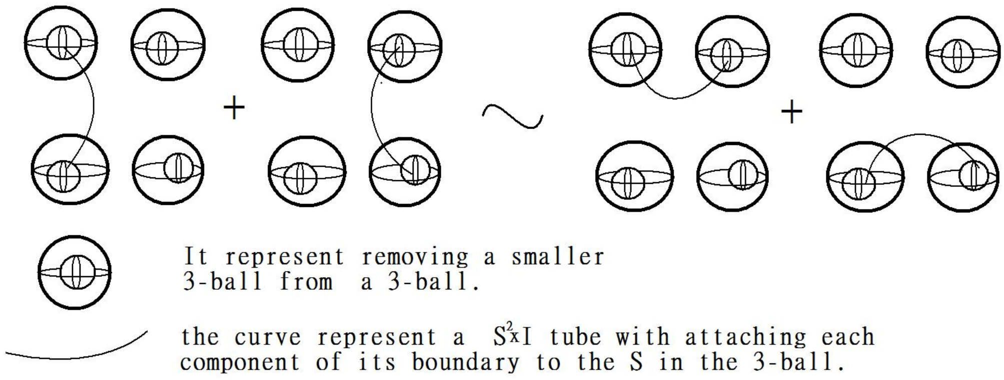

Figure 40: $New4Tu$-relation

Now we give a theorem concerning this type of Roseman move.

Theorem 5.1. If we add the relations $S^1 \times S^2 \sim 2$ and $New4Tu$-relation into the Morphisms of Cob^4, then we can find the homotopy equivalence for the first type Roseman move(bubble). It can be shown by following diagrams:

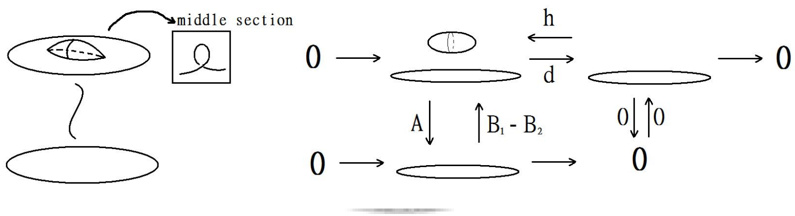

Figure 41: The chian homotopy equivalence

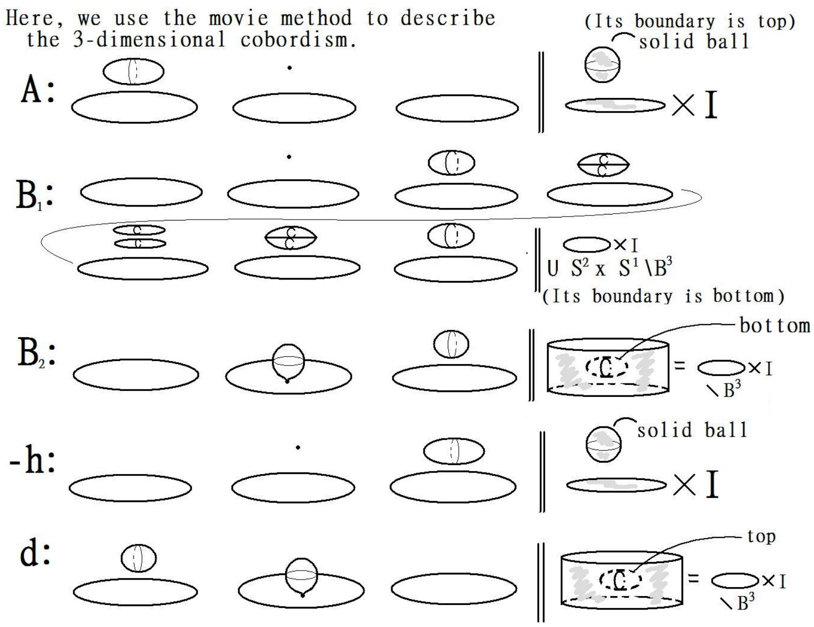

Figure 42: The maps: A, B, h and d

Proof. First, we check that whether they are chain maps or not.
The equality $d \circ B_1 - d \circ B_2 = 0$ is due to each of them are $S^2 \times S^1 \backslash B^3$ with its boundary S^2 in the boundary of $B^3 \times I$. And $(d = 0) \circ A = 0$ is obvious.
Secondly, we want to show it is a homotopy equivalence: The equality $A \circ (B_1 - B_2) = Id$ follows the relation $S^2 \times S^1 \sim 2$. Since $A \circ B_1 = A \circ B_2 \cup S^2 \times S^1$ and $A \circ B_2 = Id$, so we have $A \circ (B_1 - B_2) = 2Id - Id = Id$. To check another case $(B_1 - B_2) \circ A$, we use the $New4Tu$-relation which is like the following picture:

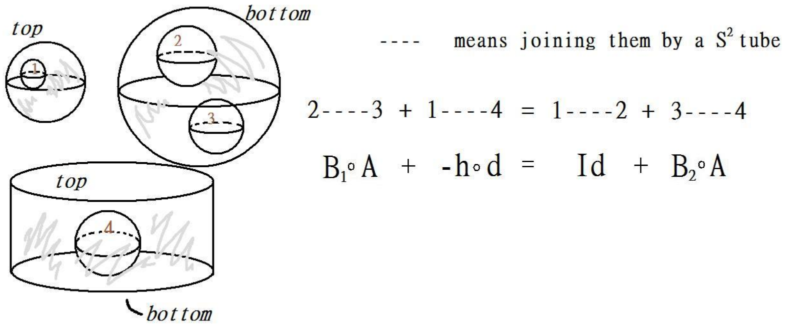

Figure 43: The $New4Tu$-relation in this case

So we get $(B_1 - B_2) \circ A - Id = h \circ d$.
Finally, we can find the equality $d \circ h = -Id$ since the ball (bottom) in the h fills the hole (top) in the d.
Hence we get a homotopy equivalence between them.

$\square$

Remark 5.2. In the $New4Tu$-relation, there is only one way to attach the S^2-tube since the 3-cobordism is embedded in R^4.

5.2 First type Roseman move (saddle)

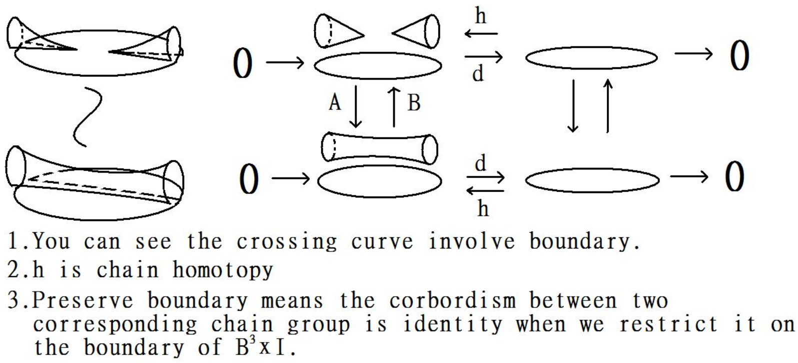

Figure 44: The first type Roseman move (saddle)

As the above picture, we encounter the first trouble which is that its crossing (curve) meet the boundary of the B^3. But it dosen't happen in the Reidemeister moves since each crossing (point) is enclosed in the boundary of a 2-disk. It will lead to many problems which let constructing the chain homotopy equivalence become more difficult. To study these problems systematically, we list some possible strategies and the difficulties we encounter. We also present some results we have found in each strategy.

In fact, there is an another problem in this case. It is that we have two choices to smooth the upper knot diagram in figure 44: One is smoothing two arcs at the same time and the other is one arc each time. But to avoid making the problem become too complicated, we first discuss the case which we smooth them at the same time.

Following list contains three approachs we consider. We try to use them to realize how we constuct the homotopy equivalence in the case (saddle).

1. *Preserve boundary*: It means we want to preserve the boundary of the corresponding surfaces which is in ∂B^3 when we construct homotopy equivalence. It leads to two situations:

(a) $h = 0$ (The chain homotopy equals zero.):

i. It won't touch the extension problem that how we can extend the local chian map to a global one.

ii. We can guess what kinds of relation we need in constructing homotopy equivalence by observing the topological type of its boundary.

iii. The difficulty will not be met now. But when we try to establish those new relations, we will find many obstructions.

(b) $h \neq 0$ (The chian homotopy doesn't equal zero.):

i. This will encounter immediately the problem that we must have some relations which involve the boundary, but we can't understand yet how to intepret this unusual phenomenon.

ii. The other problem is the extension of the chain homotopy. Since we can't determine what the manifold is outside the B^3, so we are not sure that whether the cobordism in the B^3 can be extended outside the B^3.

2. *Admit changing boundary*: The problems in this strategy is very similar to those in the strategy1 ($h \neq 0$).

(a) The relation will involve boundary.

(b) The extension problem will arise.

(c) In this section, I will present a natural way to construct the homotopy equivalence which admits changing boundary.

(d) If we permit changing boundary, then the definition of the morphisms in Cob^4 must be modified.

3. *Classify the global cases*: It means to construct the chain homotopy equivalence for each given global case.

(a) It seems that it is impossible to become a good way to solve this problem completely, since there are too many global examples and most of them are very complicated.

(b) The advantage of this strategy is that we can avoid that its crossings meet the boundary of B^3.

(c) It maybe can give us some possible chain homotopy equivalence for the local case.

Now if we adopt the strategy 2, then we can have a natural way to find a homotopy equivalence by using $S^1\times$ first Reidemeister move.

Lemma 5.3. If we add two additional relations in the Cob^4: $S^1\times S^1\times S^1 = 2$ and $S^1 \times 4Tu - relation$ which is like the following picture,

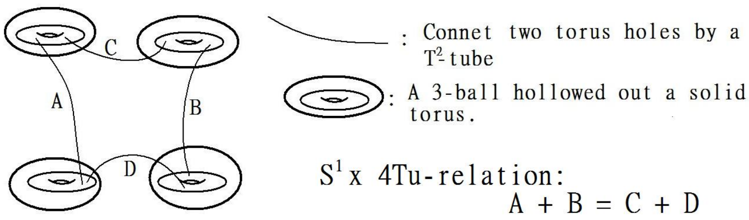

Figure 45: The $S^1 \times 4Tu - relation$

then we can have a homotopy equivalence between the following two knot diagrams.

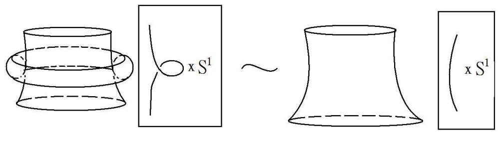

Figure 46: The two equivalent knot diagrams

Proof. We will prove it by constructing homotopy equivalence in the following diagram:

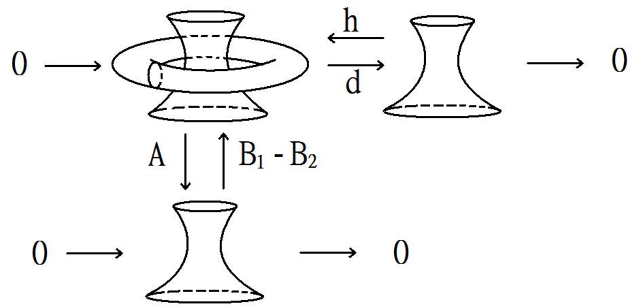

Figure 47: The chain complexes of them

The following picture describes what those maps are by *movie method*. In fact, it is just $S^1\times$ the homotopy equivalence in the first Reidemeister move.

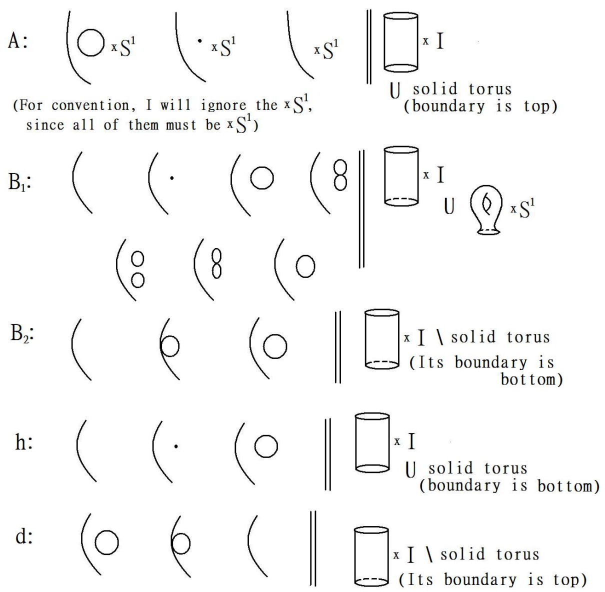

Figure 48: The homotopy equivalence

Since the cobordisms here are just $S^1\times$ those cobordisms in the first Reidemeister move, so it is not difficult to check that they are chain maps: $d\circ(B_1-B_2)=0$ and $0\circ A=0$. To check it is a homotopy equivalence we use $S^1\times S^1\times S^1=2$ which leads $(B_1-B_2)\circ A=Id$ and use $S^1\times 4Tu-relation$ which implies $A\circ(B_1-B_2)-Id=h\circ d$.
And the inquality $h\circ d=-Id$ is due to that the solid torus in h will fill the solid torus hole in d.

The following picture describes how we use the $S^1\times 4Tu$-relation in this case.

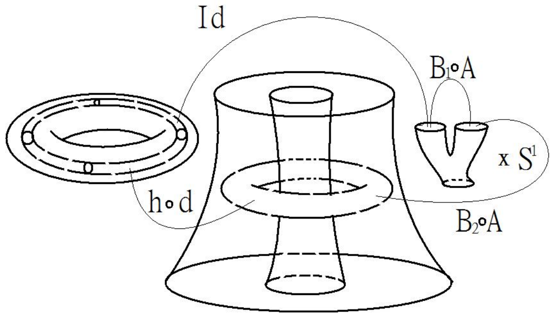

Figure 49: Apply $S^1\times 4Tu-relation$ to the case

□

On the other hand, the homotopy equivalence we construct in the section 4.1 gives us the following lemma.

Lemma 5.4. If we use the relations: $S^2\times S^1=2$ and $New4Tu$-relation on Cob^4, then there is a homotopy equivalence between the following two knot diagrams.

Now, we combine the two homotopy equivalences, and take one piece from each diagram. It is the region enclosed by the red bold rectangle. We

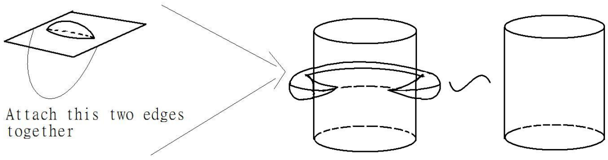

Figure 50: The two equivalent knot diagrams

can see the figure 51. Then we can obtain the homotopy equivalence of the first Roseman move (saddle), but the boundaries in these 3-cobordisms are not fixed.

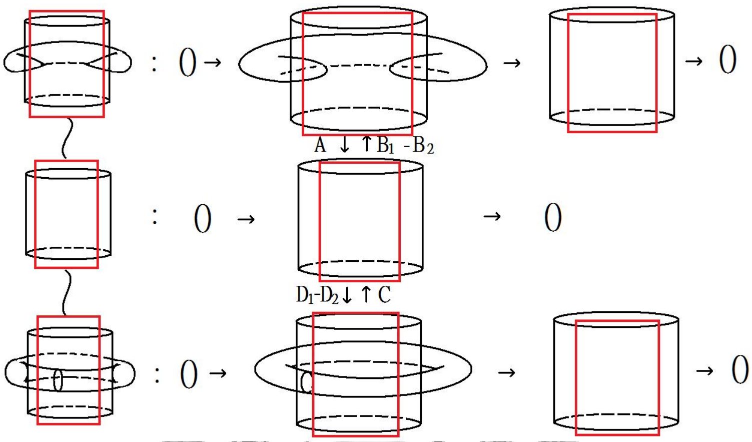

Figure 51: Cutting a piece of those surfaces

This can be stated as the following theorem.

Theorem 5.5. By adding some relations which involve boundary, we can obtain the homotopy equivalence of the first Roseman move(saddle).

Remark 5.6. 1. The relations which are used in this theorem can be gotten by observing how the cutting we did here effect the original relations: $S^2 \times S^1 = 2$, $S^1 \times S^1 \times S^1 = 2$ and the two kinds of $4Tu$-relation. Figure 52 describes two of them which are induced from the relations: $S^2 \times S^1 = 2$ and $S^1 \times S^1 \times S^1 = 2$. One is "*Two solid tori*" with their

Figure 52: New relations in the theorem

boundaries in $\partial(B^3) \times I$ and the other is "$Torus \times I$" with its boundary also in $\partial(B^3) \times I$. Those relations induced from the $S^1 \times 4Tu$-relation and $New4Tu$-relation are not stated here since it is not easy to draw.

Note that all of its relations involve boundary of $B^3 \times I$, but I don't know how to explain this phenomenon yet.

2. If we restrict the 3-cobordisms on the boundary of B^3 and remove one point on the ∂B^3 then it will become a 2-dim'l cobordisms in $B^2 \times I$ like a morphism in Cob^3. For example the boundaries of $B_1 \circ C$ and $B_2 \circ C$ are like the following picture.

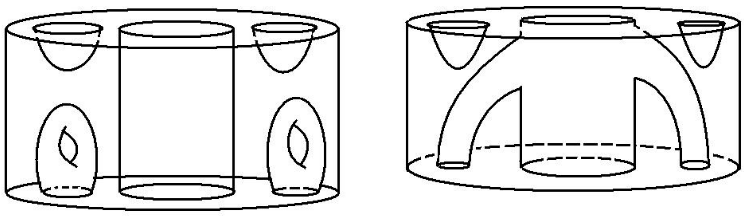

Figure 53: The shape of boundary

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