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Department of Mathematics
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Master Thesis

射影直線上的 Gromov-Witten 理論

The Gromov-Witten theory of $\mathbb{P}^1$

賴冠文
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謝辭

這篇論文雖然沒有足以為道的貢獻，卻已經讓我學到了很多。欠缺基礎的我研讀文獻的過程滿是胡亂摸索，常常以為觸及了什麼，過一陣子才驚覺自己正身陷於錯誤的理解中。跌跌撞撞的最後總算能夠約略掌握目的與動機的全貌，而這時我早已結束了論文口試、學期的課程以及法國的暑期學校兼研討會——那是一趟對我影響深遠的旅程。撰寫論文的過程短短半年，其中有太多恩人在各個層面伴演了關鍵的角色。

知識不存在於特定的個人、文章或是任何的討論、演說，它即是這一切動靜本身，獲得它的方法就是讓自己融入這一切。令我深刻體會到這點的，是兩年來不厭其煩地引導我、督促我並改變我的林惠雯教授，感謝您無論在什麼方面總是提醒我我應該重視什麼。

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摘要

在這篇文章中，我概述了關於 Gromov-Witten 不變量與 Hurwitz 數之間如何建立對應的工作，以及詳細探討 Toda 階序的 Hirota 方程。該階序能夠提供相當程度的遞迴關係以計算射影直線上的 Gromov-Witten 不變量。主要的參考文獻為 A. Okounkov 與 R. Pandharipande 的一系列論文[11, 12]。

關鍵辭：ELSV 方程，Gromov-Witten 理論，Hurwitz 數，τ-函數，Toda 階序，完備輪換，偏移對稱函數，無限維楔表示論。

Abstract

In this article, I would like to outline the work about the correspondence between Gromov-Witten invariants and Hurwitz numbers, and concentrate mainly on the detailed study of Hirota equations for the Toda hierarchy which provides certain recurrence relations for relative Gromov-Witten invariants of $\mathbb{P}^1$. The papers of A. Okounkov and R. Pandharipande [11, 12] are the main sources of my study.

Keywords: completed cycle, ELSV formula, Gromov-Witten theory, Hurwitz number, infinite wedge representation, shifted symmetric function, τ-function, Toda hierarchy.

Contents

Introduction

In the absolute Gromov-Witten theory of $\mathbb{P}^n$, the localization technique is applied to compute the 1-point invariants with descendents. For multi-point cases, the divisor relation [7] was used to reduce them to the 1-point case [6]. In the relative theory, however, the computation becomes rather difficult even for target space $\mathbb{P}^1$.

The main purpose of this article is to study some recurrence relations [11] for the stationary Gromov-Witten invariants of $\mathbb{P}^1$ relative to 0 and ∞, which is a conclusion of a general correspondence between the Gromov-Witten invariants and the Hurwitz numbers proved in the same paper.

The Hurwitz numbers enumerates the numbers of covers with assigned ramification conditions over a smooth target curve X. It has a character-theoretic expression, more precisely, an expression in the space of *shifted symmetric functions*. The definition of the Hurwitz numbers could be extended over the space of such functions, which provides a necessary platform for establishing the GW/H correspondence.

The construction of the GW/H correspondence starts with the degeneration formula with some reinterpretation of the 0-point factors by the Hurwitz numbers:

$$\left\langle \prod_{i=1}^{n} \tau_{k_i}(\omega), \eta^1, \ldots, \eta^m \right\rangle_d^{\bullet X} = \sum_{|\mu^1|=\cdots=|\mu^n|=d} \mathsf{H}_d^X(\mu^1, \ldots, \mu^n, \eta^1, \ldots, \eta^m) \prod_{i=1}^{n} \mathfrak{z}(\mu^i) \left\langle \mu^i, \tau_{k_i}(\omega) \right\rangle^{\bullet X}.$$

Then the remaining work is to show that the right hand side can be integrated to form the required (extended) Hurwitz numbers. The whole process is taken within the function space mentioned above.

However, in the above strategy, a special case is needed and is refered to [12]. The special correspondence requires the method about the equivariant technique as well as the operator formalism, in which the ELSV formula [1] plays an essential role. Actually, the ELSV formula equates the simple Hurwitz numbers with the Hodge integrals up to a multiple of constants, thus can be seen as a toefold of the connection between the Gromov-Witten theory and the Hurwitz numbers.

After the construction of the GW/H correspondence, we then get into the main part: the 2-Toda hierarchy as well as the recurrence relations for the GW-invariants of $\mathbb{P}^1$ relative to two points. As a consequence of the commutativity of specific operators, the τ-function introduced in the operator formalism satisfies a series of equations called 2-Toda hierarchy. On the other hand, a generating funciton for the relative GW-invariants will arises as a special case of the τ-function, and the recurrence relations are given in fact by the Toda hierarchy.

Organization of the paper. Section 1 starts with the definition of Gromov-Witten invariants, including a word on the disconnected theory. Next the definition of Hurwitz numbers is recalled, which will be translated into the representation-theoretical setting to lead in the notion of completed cycles. Some motivations about introducing the completed cycles are organized at the end.

Section 2, as a set-up of the later sections, is a brief summary of the operator formalism. Infinite wedge space, Murnaghan-Nakayama rule together with the notions and relations of some important operators, say, bosons, fermions and the operators $\mathcal{E}$, are the main contents of this section.

Section 3 is devoted to the GW/H correspondence, whose proof is splitted into two stages: the special correspondence and the full correspondence. The special correspondence, derived from the localization formula, is worked out essentially through the operator formalism. Here some computational details, especially on how the localization formula works, will be given; The full correspondence is a conclusion of the degeneration formula, where the special case provides as a sufficient constraint to confirm that the degeneration formula actually gives the exact answer.

Section 4 is an appication of the GW/H correspondence. First a general τ-function is introduced and is proved to satisfy the Hirota equations for the Toda hierarchy. Then we show that a generating function for the relative GW-invariants over two points of $\mathbb{P}^1$ is a special case of the τ-function, so that the Toda hierarchy would provides a recursive recipe for computing those invariants. I this part I'd try to work out the computation details that were omitted in the source paper.

1. Gromov-Witten invariants and Hurwitz numbers

1.1. **Gromov-Witten invariants.** Fix a smooth projective curve X. Let $\overline{M}_{g,n}(X,d)$ be the moduli space of genus g, degree d stable maps with n marked points and target space X. Also let $\mathrm{ev}_i : \overline{M}_{g,n}(X,d) \longrightarrow X$ be the evaluation map at the i-th marked point.

We take into account two kinds of classes in $A^1(\overline{M}_{g,n}(X,d))$. One is $\mathrm{ev}_i^*(\omega)$ with $\omega \in A^1(X)$ being the Poincaré dual of the point class, the other is the first Chern class ψ_i of the i-th cotangent bundle on $\overline{M}_{g,n}(X,d)$. The stationary Gromov-Witten invariants with descendents ("GW-invariants" for short) are defined to be

$$\left\langle \prod_{i=1}^{n} \tau_{k_i}(\omega) \right\rangle_{g,d}^{\circ X} = \int_{[\overline{M}_{g,n}(X,d)]^{vir}} \prod_{i=1}^{n} \psi_i^k \mathrm{ev}_i^*(\omega).$$

Here the upperright circle is used to emphasize that we're considering connected domain curves.

1.1.1. *The disconnected theory.* The moduli space of stable maps with possibly disconnected domain, roughly speaking, can be thought of as a Deligne-Mumford stack in the following way:

$$\overline{M}^{\bullet}_{g,[n]}(X,d) = \coprod_{\{(g_i,[n_i],d_i)\}\in \mathrm{Part}[g,[n],d]} \left(\prod_i \overline{M}_{g_i,[n_i]}(X,d_i) \right) \Big/ \mathrm{Aut}\{(g_i,[n_i],d_i)\},$$

where $[n] = \{1,\dots,n\}$, and

$$\sum_i 2g_i - 2 = 2g - 2, \quad \coprod_i [n_i] = [n], \quad \sum_i d_i = d.$$

The quotient by $\mathrm{Aut}\{(g_i,[n_i],d_i)\}$ is necessary since nontrivial stabilizers and repetition of moduli points occurs whenever there're repeated copies in the direct product. Principally, the generating function of disconnected GW-invariants over all possible data is the exponential function with variables being the connected ones.

1.1.2. *Relative Gromov-Witten invariants.* Consider a branched covering $p : C \longrightarrow X$ of degree d. We define a *profile* over a point $q \in X$ to be a partition η of d obtained from the multipilicities of $p^{-1}(q)$.

Fix m points $q_1,\dots,q_m \in X$ and partitions $\eta^1,\dots,\eta^m$ of degree d. The moduli space $\overline{M}_{g,n}(X,\eta^1,\dots,\eta^m)$ parameterizes all genus g, n-pointed relative stable maps toward X with profiles η^i at q_i, $i = 1,\dots,m$. The relative GW-invariants are defined by

$$\left\langle \prod_{i=1}^{n} \tau_{k_i}(\omega), \eta^1,\dots,\eta^m \right\rangle_{g,d}^{\circ X} = \int_{[\overline{M}_{g,n}(X,\eta^1,\dots,\eta^m)]^{vir}} \prod_{i=1}^{n} \psi_i^k \mathrm{ev}_i^*(\omega).$$

The moduli spaces $\overline{M}^{\bullet}_{g,n}(X,\eta^1,\dots,\eta^m)$ and invariants $\left\langle \prod_{i=1}^{n} \tau_{k_i}(\omega), \eta^1,\dots,\eta^m \right\rangle_{g,d}^{\bullet X}$ in the disconnected notion are similarly defined.

Finally, I would like to mention that the expected dimension of the relative moduli space, whether disconnected or not, is

$$2g - 2 + n + d(2 - 2g(X)) - \sum_{i=1}^{n} (d - l(\eta^i)),$$

where $l(\eta)$ stands for the length of the partition η.

1.2. **Hurwitz numbers.** The Hurwitz number $\mathsf{H}_d^X(\eta^1, \ldots, \eta^m)$ is defined to be the number of isomorphism classes of coverings, possibly disconnected, over X which are unbranched except over $q_1, \ldots, q_m$ with profiles $\eta^1, \ldots, \eta^m$, respectively. The countings are weighted by $1/|\mathrm{Aut}(-)|$.

In fact, for the dimension being zero, the moduli space $\overline{M}^{\bullet}_{g,0}(X, \eta^1, \ldots, \eta^m)$ with no marked point parametrizes the coverings satisfying the Riemann-Hurwitz formula

$$2g - 2 + d(2 - 2g(X)) = \sum_{i=1}^{m} (d - l(\eta^i)). \tag{1.1}$$

Hence the Hurwitz number is exactly

$$\mathsf{H}_d^X(\eta^1, \ldots, \eta^m) = \left\langle \eta^1, \ldots, \eta^m \right\rangle_d^{\bullet X}. \tag{1.2}$$

1.2.1. *Description by permutations.* For convenience let $X = \mathbb{P}^1$ at this stage.

Let S_d be the d-th symmetric group. For any partition η of degree d, we can associate it with the conjugacy class C_η in S_d of the corresponding cycle type. The Hurwitz number has the following equivalent description

$$\mathsf{H}_d^{\mathbb{P}^1}(\eta^1, \ldots, \eta^m) = \frac{|\{(s_1, \ldots, s_m) \in \prod_{i=1}^m C_{\eta_i} : s_1 \cdots s_m = 1\}|}{|S_d| = d!}.$$

This follows from the one-to-one correspondence between the isomorphism classes of Hurwitz coverings and the conjugate classes of the above m-tuples (s_i) as in the following:

Given a Hurwitz covering $C \xrightarrow{p} \mathbb{P}^1$ with profiles $\eta^1, \ldots, \eta^m$, let $C^\circ \to U$ be the associated cover of $U := \mathbb{P}^1 \backslash \{q_1, \ldots, q_m\}$. We've known that $\pi_1(U) = \langle \gamma_1, \ldots, \gamma_m : \gamma_1 \cdots \gamma_m = 1 \rangle$, where each γ_i is a generator around the punctured point q_i. For each i, consider a neighborhood V around q_i which is small enough such that p maps each connected sheet of $p^{-1}(V)$ isomorphically toward V, then the liftings of γ_i will associate a permutation s_i of the sheets over V. Thus the cover $C \to \mathbb{P}^1$ will give us an m-tuple $(s_1, \ldots, s_m)$ up to conjugation, and $\gamma_1 \cdots \gamma_m = 1$ imples $s_1 \cdots s_m = 1$. It's straightforward to see that their cycle types coincide with the original profile data. Note that an isomorphism f between two coverings $C \to \mathbb{P}^1$ and $C' \to \mathbb{P}^1$ induces a conjugation between the m-tuples (s_i) and $f^{-1}(s_i')f$ thus a conjugation between (s_i) and (s_i').

Conversely, assume an m-tuple $(s_1, \ldots, s_m) \in \prod_i C_{\eta_i}$ is given. Let $\tilde{U}$ be the universal covering of U and recall that $\pi_1(U) \cong \mathrm{Deck}(\tilde{U}/U)$. Also let

$[d] = \{1, \ldots, d\}$ be a set equipped with $\pi_1(U)$-action given by the homomorphism

$$\begin{aligned} \pi_1(U) &\longrightarrow S_d \\ \gamma_i &\longmapsto s_i. \end{aligned}$$

A Hurwitz cover of $\mathbb{P}^1$ with profiles $\eta^1, \ldots, \eta^m$ can be constructed by extending the following cover of U to $\mathbb{P}^1$:

$$\tilde{U} \times_{\pi_1(U)} [d] \longrightarrow U$$

under the equivalence relation $(x\gamma^{-1}, k) \sim (x, \gamma k)$. The isomorphism class of the resulting covering is independent of the conjugation of $(s_1, \ldots, s_m)$.

The maps constructed above are converse to each other.

Consider the S_d-action on $\prod_i C_{\eta_i}$ by conjugation: $g \cdot (s_i) = g^{-1}(s_i)g$. Let $\mathrm{Orb}(s_i)$ be the orbit and $\mathrm{Stab}(s_i)$ be the stablizer of (s_i) under the action. By the above correspondence, the countings for both sides regardless of the weights can be explicitly written down as

$$\sum_{\text{Isom. class of } C\to\mathbb{P}^1} 1 = \sum_{\substack{(s_i)\in\prod_i C_{\eta_i} \\ \prod_i s_i = 1}} \frac{1}{|\mathrm{Orb}(s_i)|}.$$

Given a covering $C \to \mathbb{P}^1$ and a corresponding (s_i), observe that, using the above construction, there's a one-to-one correspondence between the sets $\mathrm{Aut}(C/\mathbb{P}^1)$ and $\mathrm{Stab}(s_i)$. It follows that

$$\sum_{\text{Isom. class of } C\to\mathbb{P}^1} \frac{1}{|\mathrm{Aut}(C/\mathbb{P}^1)|} = \sum_{\substack{(s_i)\in\prod_i C_{\eta_i} \\ \prod_i s_i = 1}} \frac{1}{|\mathrm{Orb}(s_i)| \cdot |\mathrm{Stab}(s_i)|} = \sum_{\substack{(s_i)\in\prod_i C_{\eta_i} \\ \prod_i s_i = 1}} \frac{1}{d!},$$

which is exactly the required identity.

1.2.2. *Burnside formula.* Counting the number of the elements as above in the symmetric group allows another viewpoint from the representation theory. Recall the following standard correspondences:

‘A partition λ of d ’
$\longleftrightarrow$ ‘A conjugacy class $C_\lambda \subset S_d$ ’
$\longleftrightarrow$ ‘An irreducible representation $\rho_\lambda : S_d \to \mathrm{End}(V_\lambda)$ ’.

We'll usually abuse notations to denote by $\dim \lambda$ the dimension of V_λ and by λ the irreducible representation ρ_λ.

Let $c_\eta = \sum_{s\in C_\eta} s$. It is easy to get that

$$\mathsf{H}_d^{\mathbb{P}^1}(\eta^1, \ldots, \eta^m) = \frac{1}{d!}[1_{S_d}] \prod c_{\eta^i} = \frac{1}{(d!)^2} \mathrm{Tr}_{\mathbb{Q}S_d}\Big(\prod c_{\eta^i}\Big), \tag{1.3}$$

where $[1_{S_d}]$ stands for the operation to capture the coefficient of the identity element and $\mathrm{Tr}_{\mathbb{Q}S_d}$ means the trace for the regular representation.

Now given an irreducible representation λ, for each partition η, let $\mathbf{f}_\eta(\lambda)$ be the eigenvalue of $\rho_\lambda(c_\eta)$. By the Wedderburn's structure theorem $\mathbb{Q}S_d \simeq \bigoplus_\lambda V_\lambda^{\oplus \dim\lambda}$, where λ runs through all non-equivalent irreducible representations of S_d.

$$\mathrm{Tr}_{\mathbb{Q}S_d}(\prod_{i=1}^m c_{\eta^i}) = \sum_\lambda (\dim\lambda)\,\mathrm{Tr}_{V_\lambda}\left(\prod_{i=1}^m c_{\eta^i}\right) = \sum_\lambda (\dim\lambda)^2 \prod_{i=1}^m \mathbf{f}_{\eta^i}(\lambda).$$

We plug this into (1.3) to yield

$$\mathsf{H}_d^{\mathbb{P}^1}(\eta^1,\dots,\eta^m) = \sum_{|\lambda|=d}\left(\frac{\dim\lambda}{d!}\right)^2 \prod_{i=1}^m \mathbf{f}_{\eta^i}(\lambda).$$

So the Hurwitz numbers can be expressed in terms of functions of partitions under the Fourier transform:

$$Z_d \longrightarrow \mathbb{Q}^{\mathscr{P}(d)}: \quad c_\eta \mapsto \mathbf{f}_\eta,$$

where Z_d is the center of $\mathbb{Q}S_d$ and $\mathscr{P}(d)$ stands for the set of partitions of d.

What we've proved is a special case of the *Burnside formula*:

$$\mathsf{H}_d^X(\eta^1,\dots,\eta^m) = \sum_{|\lambda|=d}\left(\frac{\dim\lambda}{d!}\right)^{2-2g(X)} \prod_{i=1}^m \mathbf{f}_{\eta^i}(\lambda). \tag{1.4}$$

Having this in hand, we can generalize the definition of the Hurwitz numbers. It is done by first generalizing $\mathbf{f}_\eta$ to be a function on $\mathscr{P} = \bigoplus_{d\geq 0}\mathscr{P}(d)$. Note that

$$\mathbf{f}_\eta(\lambda) = \frac{\mathrm{Tr}_{V_\lambda}(c_\eta)}{\dim\lambda} = \frac{\sum_{s\in C_\eta}\mathrm{Tr}_{V_\lambda}(s)}{\dim\lambda} = |C_\eta|\frac{\chi_\eta^\lambda}{\dim\lambda}$$

where χ_η^λ is the value of the character of λ at the class C_η. Thus an extended function $\mathbf{f}_\eta \in \mathbb{Q}^{\mathscr{P}}$ can be defined by

$$\mathbf{f}_\varnothing \equiv 1; \qquad \mathbf{f}_\eta(\lambda) = 0 \quad \text{if } |\eta| > |\lambda| \tag{1.5}$$

$$\mathbf{f}_\eta(\lambda) = \binom{|\lambda|}{|\eta|}|C_\eta|\frac{\chi_\eta^\lambda}{\dim\lambda} \quad \text{if } 0 < |\eta| \leq |\lambda|.$$

Hence the Fourier transform can be extended to

$$\bigoplus_{d\geq 0} Z_d \longrightarrow \mathbb{Q}^{\mathscr{P}}: \quad c_\eta \mapsto \mathbf{f}_\eta. \tag{1.6}$$

And the Hurwitz number can be generalized to profiles with arbitrary degrees keeping the form (1.4).

Remark. Given any partition μ, we denote by $m_i(\mu)$ the number of i's appearing in μ. Suppose there're partitions $\eta^1, \ldots, \eta^m$ with degree $\leq d$. Let $\boldsymbol{\eta}^1, \ldots, \boldsymbol{\eta}^m$ be the partitions obtained from $\eta^1, \ldots, \eta^m$ by adding 1's until all the degrees are equal to d. Then by using (1.5) as well as a direct computation, we can get

$$\mathsf{H}^X_d(\eta^1, \ldots, \eta^m) = \left[\prod_{i=1}^m \binom{m_1(\boldsymbol{\eta}^i)}{m_1(\eta^i)}\right] \mathsf{H}^X_d(\boldsymbol{\eta}^1, \ldots, \boldsymbol{\eta}^m).$$

1.3. **Completed cycles.** The shifted action of S_d on $\mathbb{Q}[x_1, \ldots, x_d]$ is by permuting the shifted variables $x_i - i$, $i = 1, \ldots, d$. Let

$$\Lambda^*(d) = \mathbb{Q}[x_1, \ldots, x_d]^{*S_d}$$

be the invariant subalgebra of the shifted action. There's a natural map $\Lambda^*(d) \to \Lambda^*(d-1)$ by setting the last variable x_d to be zero. The algebra of shifted symmetric functions Λ^* is obtained by taking the projective limit

$$\Lambda^* = \varprojlim_d \Lambda^*(d).$$

It's easy to see that $\Lambda^* \subset \mathbb{Q}^{\mathscr{P}}$.

Define the *completed cycle* for each $k \in \mathbb{N} \cup \{0\}$ by

$$\mathbf{p}_k(x) = k![z^k]\mathbf{e}(x, z), \quad \mathbf{e}(x, z) = \sum_{i=1}^{\infty} e^{z(x_i - i + \frac{1}{2})}, \tag{1.7}$$

where $[z^k]$ means we only take the coefficient of z^k. However, buy directly expanding it one might get

$$\mathbf{p}_k(x) \text{ “ = ” } \sum_{i=1}^{\infty} (x_i - i + \frac{1}{2})^k,$$

which doesn't belong to the algebra Λ^*! The correct functions must be obtained by taking the Riemann ζ-function regularization:

$$\mathbf{p}_k(x) = \sum_{i=1}^{\infty} [(x_i - i + \frac{1}{2})^k - (-i + \frac{1}{2})^k] + (1 - 2^{-k})\zeta(-k).$$

Such shifted symmetric power sums canonically form a set of generators for Λ^*:

$$\Lambda^* = \mathbb{Q}[\mathbf{p}_1, \mathbf{p}_2, \ldots].$$

By [5], the image of the Fourier transform (1.6) lies in Λ^*. Moreover there's a transformation between those $\mathbf{f}$'s and $\mathbf{p}$'s:

$$\mathbf{f}_\mu = \frac{1}{\prod \mu_i} \mathbf{p}_\mu + \text{(lower degree terms)}, \quad \mathbf{p}_\mu := \prod_{i=1}^{l(\mu)} \frac{\mathbf{p}_{\mu_i}}{\mu_i}. \tag{1.8}$$

In particular $\mathbf{f}_{(2)} = \frac{\mathbf{p}_2}{2!}$. Hence the Fourier transform arises as an isomorphism between $\bigoplus_{d\geq 0} Z_d$ and Λ^*. Furthermore, using the expression (1.4)the Hurwitz number can be extended as a function on Λ^*, especially bringing into the completed cycles would be the backbone of the following discussions.

1.3.1. *Why completed cycles?* Consider the smooth locus $M^\bullet_{g,n}(X,d) \overset{\iota}{\hookrightarrow} \overline{M}^\bullet_{g,n}(X,d)$. A correspondence was found by a quite direct manner in Proposition 1.1 [11]:

$$\int_{[M^\bullet_{g,n}(X,d)]} \prod_{i=1}^{n} \tau_{k_i}(\omega) = \frac{1}{\prod k_i!} \mathsf{H}^X_d((k_1+1),\ldots,(k_n+1)).$$

Note that $[M^\bullet_{g,n}(X,d)] = \iota^*[\overline{M}^\bullet_{g,n}(X,d)]^{vir}$ (see Proposition 5.2 [10]). Therefore

$$\begin{aligned}\left\langle \prod_{i=1}^{n} \tau_{k_i}(\omega) \right\rangle^{\bullet X}_d &= \frac{1}{\prod k_i!} \mathsf{H}^X_d((k_1+1),\ldots,(k_n+1)) + \triangle \\ &= \sum_{|\lambda|=d} \left(\frac{\dim\lambda}{d!}\right)^{2-2g(X)} \prod_{i=1}^{n} \mathbf{f}_{(k_i+1)}(\lambda) + \triangle,\end{aligned}$$

where $\triangle$ stands for the contribution from the boundary divisors.

The defect can be modified by considering the completed cycles $\{\mathbf{p}_k\}$. For instance, when $X = \mathbb{P}^1$ the following *special GW/H correspondence* will be introduced in Section 3:

$$\left\langle \prod_{i=1}^{n} \tau_{k_i}(\omega) \right\rangle^{\bullet \mathbb{P}^1}_d = \sum_{|\lambda|=d} \left(\frac{\dim\lambda}{d!}\right)^{2} \prod_{i=1}^{n} \frac{\mathbf{p}_{k_i+1}(\lambda)}{(k_i+1)!}. \tag{1.9}$$

We can see that completing the cycles corresponds to including the boundary strata of the moduli space.

If we denote by $\overline{(k)}$ the preimage of $\mathbf{p}_k/k$ under the Fourier transform (1.6), the right hand side of (1.9) can be rewritten as

$$\frac{1}{\prod k_i!} \mathsf{H}^{\mathbb{P}^1}_d(\overline{(k_1+1)},\ldots,\overline{(k_n+1)})$$

under the extended notion of Hurwitz numbers. This is a "completion" of the classical Hurwitz theory. In this modern fashion the Hurwitz theory would become more accessible in combinatorial sense. One known reason is that the completed cycles could be naturally manipulated in the operator formalism of the infinite wedge representation.

Remark. From (1.8) we could only know that the transformation matrix between the two basis $\{\mathbf{f}_\mu\}$ and $\{\mathbf{p}_\mu\}$ is triangular. The explicit formulae for computing the *completion coefficients* for expressing the completed cycles in $\{\mathbf{f}_\mu\}$ are given by Proposition 1.6 and 3.2, [11], where Proposition 1.6, as a conclusion of the GW/H correspondence, expresses the coefficients in some 1-point relative invariants that would reveal some geometric meanings about the coefficients, while Proposition 3.2 is obtained from Proposition 1.6 by the operator formalism and serves as a down-to-number solution.

A further geometric interpretation of the completed cycles could be found in [13].

2. The operator formalism

In this section a brief summary of the operator formalism is given.

The first part is about Fock representations, in which we'll take a glance at the bosonic Fock space, and then focus on the infinite wedge space (fermionic Fock space). The later one will be our operating platform for deriving the special GW/H correspondence and the main results. Some fundamental operators, for example, bosons and fermions, will be introduced in this part.

Next is a discussion about the equivalence of the above two Fock representations. The equivalence would allow us to write the fermions in terms of bosons, which is essential to the Hirota equations for the Toda hierarchy in Section 4. A brief introduction of the Murnaghan-Nakayama rule is also included, which plays an important role in translating the operator language into the language of characters and vice versa.

The final part is an introduction to the operators $\mathcal{E}$, which can be seen as a generalization of the bosons and will be frequently used in the later sections.

Besides the source paper [11], the main reference for this section is [8]. The part about fermions in bosons is refered to Chapter 14, [4].

2.1. **The Fock representations.**

2.1.1. *Bosonic Fock space.* Let R_B be a $\mathbb{C}$-algebra generated by symbols $\{\alpha_n : n \in \mathbb{Z}\backslash\{0\}\}$, called *bosons*, with relations given by the commutators:

$$[\alpha_l, \alpha_r] = l\delta_{l+r,0}. \tag{2.1}$$

We define the bosonic Fock space to be $\mathfrak{B} = \mathbb{C}[x_1, x_2, \ldots; q, q^{-1}]$. The following representation

$$\begin{aligned} \rho_B : R_B &\longrightarrow \mathrm{End}(\mathfrak{B}) \\ \alpha_n &\longmapsto \frac{\partial}{\partial x_n}, \quad n > 0 \\ \alpha_{-n} &\longmapsto n x_n, \quad n > 0 \end{aligned}$$

is called the bosonic Fock representation.

2.1.2. *Infinite wedge space (Fermionic Fock space).* Given *fermions* ψ_k, ψ_k^* indexed by the half-integers, i.e. $k \in \mathbb{Z}+\frac{1}{2}$, let R_F be the $\mathbb{C}$-algebra generated by them with the following anti-commutation relations

$$\{\psi_i, \psi_j\} = \{\psi_i^*, \psi_j^*\} = 0, \quad \{\psi_i, \psi_j^*\} = \delta_{ij}.$$

In what follows, I'll introduce the notion of the *Infinite wedge space* (or *Fermionic Fock space*), which serves as a representation space for the fermionic algebra.

Consider subsets $S = \{s_1 > s_2 > s_3 > \cdots\} \subset \mathbb{Z} + \frac{1}{2}$ satisfying:

(i) $S_+ := S \backslash (\mathbb{Z} + \frac{1}{2})_{<0}$ is finite,
(ii) $S_- := (\mathbb{Z} + \frac{1}{2})_{<0} \backslash S$ is finite.

Denote by $|S\rangle$ the following infinite wedge product:

$$|S\rangle = \underline{s_1} \wedge \underline{s_2} \wedge \underline{s_3} \wedge \ldots.$$

then the infinite wedge space is defined to be the following infinite dimensional vector space

$$\mathfrak{F} = \bigoplus_S \mathbb{C}\,|S\rangle.$$

In this fermionic case, we also consider the dual space $\mathfrak{F}^*$ with dual elements denoted by $\langle S|$. The inner product on $\mathfrak{F}$ is defined by the conditions $\langle S|S'\rangle = \delta_{S,S'}$ for all S, S'.

The fermionic Fock representation is defined by

$$\begin{aligned} \rho_F : R_F &\longrightarrow \mathrm{End}(\mathfrak{F}) \\ \psi_k &\longmapsto \underline{k}\wedge \end{aligned}$$

while $\rho_F(\psi_k^*)$ is defined to be the adjoint of $\rho_F(\psi_k)$ with respect to the inner product. Such operation is the same as contracting $\underline{k}$ from the left. For simplicity, the symbol $\rho_F(\quad)$ will be omitted later if there's no confusion caused.

2.1.3. *Charge and energy.* Define the normal ordering $:\cdot:$ by

$$:\psi_i\psi_j^*:=\begin{cases}\psi_i\psi_j^*, & j>0\\ -\psi_j^*\psi_i, & j<0\end{cases}$$

and let $E_{i,j} =: \psi_i\psi_j^* :$. The charge and the Hamiltonian operators are defined to be

$$C=\sum_{k\in\mathbb{Z}+\frac{1}{2}} E_{k,k} \quad \text{and} \quad \mathscr{H}=\sum_{k\in\mathbb{Z}+\frac{1}{2}} kE_{k,k}$$

respectively. An eigenvalue of any eigenstate for C would be called *charge* of this state; Similarly an eigenvalue for $\mathscr{H}$ would be called *energy*. For instance, by acting them on arbitrary state $|S\rangle$ of the basis for $\mathfrak{F}$, we can get

$$C\,|S\rangle = (|S_+|-|S_-|)\,|S\rangle \quad \text{and} \quad \mathscr{H}\,|S\rangle=\left(\sum_{s\in S_+} s-\sum_{s\in S_-} s'\right)|S\rangle\,.$$

Thus the elements in the basis have definite charges and energies. It follows that the infinite wedge space $\mathfrak{F}$ can be decomposed into subspaces $\mathfrak{F}_l$ consisting of states with charge $l\in\mathbb{Z}$, and each $\mathfrak{F}_l$ can be further decomposed with respect to the energies, say,

$$\mathfrak{F}=\bigoplus_{\text{charge } l\in\mathbb{Z}} \mathfrak{F}_l=\bigoplus_{\substack{\text{charge } l\in\mathbb{Z}\\ \text{energy } d\geq\frac{l^2}{2}}} \mathfrak{F}_l^d.$$

The condition $d\geq\frac{l^2}{2}$ comes from the fact that for each $\mathfrak{F}_l$, there's a unique ground state

$$|l\rangle=\underline{l-\frac{1}{2}}\wedge\underline{l-\frac{3}{2}}\wedge\underline{l-\frac{5}{2}}\wedge\ldots,$$

which has energy $l^2/2$. Especially,

$$|0\rangle=\underline{-\frac{1}{2}}\wedge\underline{-\frac{3}{2}}\wedge\underline{-\frac{5}{2}}\wedge\ldots,$$

called the *vacuum state*, is the ground state of $\mathfrak{F}_0$. For any operator X on $\mathfrak{F}$, we'll call $\langle X\rangle := \langle 0\,|X|\,0\rangle$ the *vacuum expectation* for X.

It's easy to check that $\mathfrak{F}_0$, as the kernal of C, is spanned by the states

$$|\lambda\rangle=\underline{\lambda_1-\frac{1}{2}}\wedge\underline{\lambda_2-\frac{3}{2}}\wedge\underline{\lambda_3-\frac{5}{2}}\wedge\ldots=\bigwedge_{i=1}^{\infty}\underline{\lambda_i-i+\frac{1}{2}}$$

indexed by all partitions $\lambda=(\lambda_1\geq\lambda_2\geq\ldots\geq 0=\ldots)$, and

$$\mathscr{H}\,|\lambda\rangle=|\lambda|\,|\lambda\rangle\,.$$

Generally the subspace $\mathfrak{F}_l$ is generated by the states

$$|\lambda, l\rangle = \underline{\lambda_1 + l - \frac{1}{2}} \wedge \underline{\lambda_2 + l - \frac{3}{2}} \wedge \underline{\lambda_3 + l - \frac{5}{2}} \wedge \dots$$

with

$$\mathscr{H} \, |\lambda, l\rangle = \left(|\lambda| + \frac{l^2}{2}\right) |\lambda, l\rangle \, .$$

2.2. **Boson-Fermion correspondence.**

2.2.1. *Bosons in Fermions.* The bosons can be realized in the infinite wedge space. Define

$$B_n = \sum_{k \in \mathbb{Z} + \frac{1}{2}} E_{k-n,k}, \quad n \in \mathbb{Z} \backslash \{0\}.$$

By the fundamental identities among commutators and anti-commutators

$$[AB, C] = A\{B, C\} - \{A, C\}B \tag{2.2}$$

$$= A[B, C] + [A, C]B, \tag{2.3}$$

one can check from (2.2) that

$$[B_n, \psi_k] = \psi_{k-n}, \quad [B_n, \psi_k^*] = -\psi_{k+n}^*$$

and then from (2.3) that

$$[B_n, B_m] = n\delta_{n+m,0},$$

hence there's a well-defined homomorphism

$$\begin{aligned} \rho_F : R_B &\longrightarrow \operatorname{End}(\mathfrak{F}) \\ \alpha_n &\longmapsto B_n. \end{aligned}$$

It's also straitforward to check in the same way that

$$[\mathscr{H}, B_{-n}] = nB_{-n}, \quad n \in \mathbb{Z},$$

which means the operation of B_{-n} would increase the energy of a state by n.

2.2.2. *Equivalence of Fock representations.* The Fock representations ρ_B and ρ_F are actually isomorphic to each other. An isomorphism can be constructed explicitly as the following:

First form the formal sum of B_n in indeterminants $\{x_1, x_2, \dots\}$

$$B(x) = \sum_{n=1}^{\infty} x_n B_n,$$

then define Φ to be a linear map from $\mathfrak{F} = \bigoplus_S \mathbb{C} \, |S\,\rangle$ to $\mathfrak{B} = \mathbb{C}[x_1, x_2, \dots; q, q^{-1}]$ by

$$\Phi(|S\,\rangle) = \sum_{l \in \mathbb{Z}} q^l \, \langle l| \, e^{B(x)} \, |S\,\rangle \, .$$

The following theorem shows that Φ is an isomorphism of representations. The isomorphicity would allows us to abuse the symbol α_n, rather than B_n, to denote a boson in the infinite wedge space $\mathfrak{F}$.

Theorem 2.1. *The morphism Φ is an isomorphism between the Fock representations ρ_B and ρ_F. In particular, for $n > 0$,*

$$\Phi B_n \Phi^{-1} = \frac{\partial}{\partial x_n}, \quad \Phi B_{-n} \Phi^{-1} = n x_n.$$

Proof. We first show that

$$\Phi(B_n |S\rangle) = \begin{cases} \frac{\partial}{\partial x_n}\Phi(|S\rangle) & \text{if } n > 0, \\ -n x_{-n}\Phi(|S\rangle) & \text{if } n < 0. \end{cases}$$

Since B_n commutes with each other for $n > 0$, we have $\frac{\partial}{\partial x_n} e^{B(x)} = e^{B(x)} B_n$. Then the first equality is obtained from

$$\Phi(B_n |S\rangle) = \sum_{l\in\mathbb{Z}} q^l \langle l| e^{B(x)} B_n |S\rangle = \sum_{l\in\mathbb{Z}} q^l \langle l| \frac{\partial}{\partial x_n} e^{B(x)} |S\rangle = \frac{\partial}{\partial x_n}\Phi(|S\rangle).$$

For $n < 0$, we need the following useful formula

$$e^A B e^{-A} = B + [A, B] + \frac{1}{2!}[A, [A, B]] + \frac{1}{3!}[A, [A, [A, B]]] + \cdots . \tag{2.4}$$

The commutation relation $[B(x), B_n] = -n x_n$ together with (2.4) yields

$$e^{B(x)} B_n e^{-B(x)} = B_n - n x_n.$$

Since B_m with $m > 0$ is an annihilator for the ground states, $\langle l| B_n = 0$. Therefore we have

$$\Phi(B_n |S\rangle) = \sum_{l\in\mathbb{Z}} q^l \langle l| e^{B(x)} B_n |S\rangle = \sum_{l\in\mathbb{Z}} q^l \langle l| e^{B(x)} B_n e^{-B(x)} e^{B(x)} |S\rangle$$
$$= \sum_{l\in\mathbb{Z}} q^l \langle l| (B_n - n x_n) e^{B(x)} |S\rangle = -n x_{-n}\Phi(|S\rangle),$$

as required.

Now we turn to prove that Φ is an isomorphism of vector spaces, which will complete the proof of the theorem by combining with the above result.

Because $B(x)$ annihilates all the ground states, we have $\Phi(|l\rangle) = q^l$. Using the previous result, any monomials of $\mathfrak{B}$ has a preimage in $\mathfrak{F}$, say,

$$\Phi(\prod_i \frac{B_{-n_i}}{n_i} |l\rangle) = q^l \prod_i x_{n_i},$$

so the morphism Φ is surjective.

The bijectivity is proved by counting dimensions: The bosonic Fock space can be rewritten as $\mathfrak{B} = \bigoplus_{l\in\mathbb{Z}} q^l \mathbb{C}[x_1, x_2, \ldots]$, in which we assign a q^l-homogeneous element a *charge l*. By definition those B_n's leave the charge unchanged under action, so the map Φ would preserve charges under the assignments.

On the other hand, each x_n could be assigned a *weight* n, hence there's a decomposition

$$\mathbb{C}[x_1, x_2, \ldots] = \bigoplus_{n} \bigoplus_{|\lambda|=n} \langle x_{\lambda_1} \cdots x_{\lambda_n} \rangle_{\mathbb{C}} \,,$$

where the λ runs through all partitions with prescript degree.

Now let's focus on the charge-l level of both representation spaces. In this case the map Φ has a more simple form

$$\begin{aligned} \Phi\big|_{\mathfrak{F}_l} : \mathfrak{F}_l &\longrightarrow \mathfrak{B}_l = q^l \mathbb{C}[x_1, x_2, \ldots] \\ |S\rangle &\longmapsto q^l \langle l| \, e^{B(x)} \, |S\rangle \,, \end{aligned}$$

from which and the fact that B_n with $n > 0$ would lower down the energy by n, it can be observed that, to an energy-d state, the survived monomials under the map must be of weight $d - \frac{l^2}{2}$. Therefore the map Φ can be further decomposed with respect to the energy

$$\Phi\big|_{\mathfrak{F}_l^d} : \mathfrak{F}_l^d \longrightarrow \mathfrak{B}_l^d := q^l \bigoplus_{|\lambda|=d-\frac{l^2}{2}} \langle x_{\lambda_1} \cdots x_{\lambda_n} \rangle_{\mathbb{C}} \,.$$

The end of Section 2.1.3 indicates that the basis of $\mathfrak{F}_l^d$ is parametrized by partitions with degree $d - \frac{l^2}{2}$, hence the dimension of $\mathfrak{F}_l^d$ is exactly the same as $\mathfrak{B}_l^d$. Since Φ is surjective, it must be an isomorphism. □

2.2.3. *Murnaghan-Nakayama rule.* Here's a useful formula connecting the theory of symmetric functions and the operator formalism.

Given partitions λ and ν of the same degree, starting with the state $|\lambda\rangle$, one may wonder what will be left after a sequence of operations by $\prod_{i=1}^{l(\nu)} \alpha_{\nu_i}$. By the energy constraint the result must be some multiple of the vacuum state $|0\rangle$. But what's the number?

Recall that any partition, thus any zero-charge state, corresponds uniquely to a Young diagram. Rearrange the α_{ν_i}'s in the product $\prod_{i=1}^{l(\nu)} \alpha_{\nu_i}$ such that the integers ν_i range from small to large from left to right, then apply them one-by-one to the state $|\lambda\rangle$. It can be found that the manipulation on the corresponding Young diagrams is exactly the recursive process for calculating the character χ_ν^λ by the Murnaghan-Nakayama rule! Hence

$$\prod_{i=1}^{l(\nu)} \alpha_{\nu_i} \, |\lambda\rangle = \chi_\nu^\lambda \, |0\rangle \,.$$

For more detail see [3].

What will be frequently used later is the dual form

$$\prod_{i=1}^{l(\nu)} \alpha_{-\nu_i} \left|0\right\rangle = \sum_{|\lambda|=d} \chi_\nu^\lambda \left|\lambda\right\rangle . \tag{2.5}$$

2.2.4. *Fermions in Bosons.* The problem of realizing fermions in bosons is more subtle. Let $z = (z, z^2, z^3, \ldots)$, the realization is attained by realizing the following generating functions for fermions

$$\psi(z) = \sum_{k\in\mathbb{Z}+\frac{1}{2}} z^{k+\frac{1}{2}}\psi_k, \quad \psi^*(z) = \sum_{k\in\mathbb{Z}+\frac{1}{2}} z^{-k-\frac{1}{2}}\psi_k^*$$

in terms of two kinds of operators:

The first is the *translation operator* T

$$T\,\underline{s_1} \wedge \underline{s_2} \wedge \underline{s_3} \wedge \ldots = \underline{s_1+1} \wedge \underline{s_2+1} \wedge \underline{s_3+1} \wedge \ldots .$$

It's trivial that

$$T\psi_k T^{-1} = \psi_{k+1}, \quad T\psi_k^* T^{-1} = \psi_{k+1}^*,$$

so that $[C, T^n] = nT^n$, therefore the charge decomposition can be rewritten as

$$\mathfrak{F} = \bigoplus_{l\in\mathbb{Z}} T^l \mathfrak{F}_0.$$

The same phenomenon can be revealed in the bosonic Fock space by directly checking that $\Phi T \Phi^{-1} = q$.

The second is the *vertex operators*

$$\Gamma_\pm(t) = \exp\left(\sum_{n>0} \frac{t_n \alpha_{\pm n}}{n}\right)$$

with $\{t_1, t_2, \ldots\}$ a sequence of indeterminants. Since the bosonic operators α_n with $n > 0$ annihilate the ground states, we have

$$\Gamma_+(t) \left|l\right\rangle = \left|l\right\rangle . \tag{2.6}$$

Also observe that $\Gamma_\pm^* = \Gamma_\mp$ and

$$\Gamma_+(t)\Gamma_-(s) = e^{\sum \frac{t_n s_n}{n}} \Gamma_-(s)\Gamma_+(t). \tag{2.7}$$

By definition we have

$$[\alpha_n, \psi(z)] = z^n\psi(z), \quad [\alpha_n, \psi^*(z)] = -z^n\psi^*(z), \tag{2.8}$$

which implies that

$$\begin{aligned} &\Gamma_\pm(t)\psi(z) = \gamma(z^{\pm 1}, t)\psi(z)\Gamma_\pm(t) \\ &\Gamma_\pm(t)\psi^*(z) = \gamma(z^{\pm 1}, -t)\psi^*(z)\Gamma_\pm(t), \quad \gamma(z,t) = \exp\left(\sum_{n\geq 1} \frac{t_n z^n}{n}\right). \end{aligned} \tag{2.9}$$

Proposition 2.2. *$\psi(z)$ and $\psi^*(z)$ can be formulated as*

$$\begin{aligned}\psi(z) &= z^C T\Gamma_-(z)\Gamma_+(-z^{-1}) \\ \psi^*(z) &= T^{-1}z^{-C}\Gamma_-(-z)\Gamma_+(z^{-1}),\end{aligned} \tag{2.10}$$

where the operator z^C acts on the charge-l subspace by multiplying z^l.

The proof is proceeded in bosonic Fock space. Let

$$\Psi(z) = \Phi\psi(z)\Phi^{-1}, \quad \Psi^*(z) = \Phi\psi^*(z)\Phi^{-1}$$

be the corresponding elements of the generating function for fermions. The commutation relation (2.8) can be rewritten as

$$[x_n, \Psi(z)] = \frac{z^{-n}}{n}\Psi(z) \text{ and } [\frac{\partial}{\partial x_n}, \Psi(z)] = z^n\Psi(z).$$

Lemma 2.3. *Let $D : \mathbb{C}[x_1, x_2, \ldots] \to \mathbb{C}[[x_1, x_2, \ldots]]$ be a differential operator, namely,*

$$D = \sum_{r\geq 0}\sum_{1\leq i_1\leq\ldots\leq i_r} P_{i_i\ldots i_r}\frac{\partial}{\partial x_{i_1}}\cdots\frac{\partial}{\partial x_{i_r}}, \quad P_{i_i\ldots i_r} \in \mathbb{C}[[x_1, x_2, \ldots]].$$

(i) *If $[x_i, D] = \lambda_i D$ for $i = 1, 2, \ldots$, then $D = D(1)\exp\left(-\sum_i \lambda_i\frac{\partial}{\partial x_i}\right)$.*
(ii) *If $[\frac{\partial}{\partial x_i}, D] = \mu_i D$ for $i = 1, 2, \ldots$, then $D(1) = c\exp\left(\sum_i \mu_i x_i\right)$, $c \in \mathbb{C}$.*

Proof. Note that, for any $f \in \mathbb{C}[x_1, x_2, \ldots]$, the operator $\exp\left(-\sum_i \lambda_i\frac{\partial}{\partial x_i}\right) =: T_\lambda$ is the parallel translation:

$$(T_\lambda f)(x_1, x_2, \ldots) = f(x_1 + \lambda_1, x_2 + \lambda_2, \ldots).$$

For (i), we replace D by DT_λ, then the statement is equivalent to that $[x_i, D] = 0$ implies $D = D(1)$, $i = 1, 2, \ldots$, which is obvious.

For (ii), we replace D by $\exp\left(-\sum_i \mu_i x_i\right) D$, then the statement is equivalent to the one: $[\frac{\partial}{\partial x_i}, D] = 0$ implies $D(1) \equiv const.$, $i = 1, 2, \ldots$, which is also obvious. □

According to the fact that any linear operator on $\mathbb{C}[x_1, x_2, \ldots]$ is a differential operator, the above lemma would implies that

$$\Psi(z) = \hat{c}\exp\left(\sum_n z^n x_n\right)\exp\left(-\sum_n \frac{z^{-n}}{n}\frac{\partial}{\partial x_n}\right),$$

where $\hat{c}$ is an operator independent of those x_n's. By definition $\psi(z)$ lifts the charge by 1, so $\hat{c} = c(z)q$ with $c(z)$ some operator depending only on the variable z.

Translating it back into the fermionic side $\mathfrak{F}$ we have

$$\psi(z) = c(z)T\Gamma_-(z)\Gamma_+(-z^{-1}).$$

To figure out what $c(z)$ is we directly act $\psi(z)$ on $|l\rangle$ by the above form as well as by definition and then compare the lowest energy terms

$$\psi(z)\,|l\rangle \begin{array}{l} = c(z)T\Gamma_-(z)\Gamma_+(-z^{-1})\,|l\rangle = c(z)\,|l+1\rangle + \text{(higher energy terms)} \\ \overset{def}{=} \sum_{k\in\mathbb{Z}+\frac{1}{2}} z^{k+\frac{1}{2}}\psi_k\,|l\rangle = z^{l+1}\,|l+1\rangle + \text{(higher energy terms)}. \end{array}$$

Hence $c(z) = z^C$ and $\psi(z)$ has the required expression.

The proof for $\psi^*(z)$ is similar and omitted here.

2.3. **The operators $\mathcal{E}$.** Let $\varsigma(z) = e^{z/2} - e^{-z/2}$. For $l \in \mathbb{Z}$ the operator $\mathcal{E}_l(z)$ is defined by

$$\mathcal{E}_l(z) = \sum_{k\in\mathbb{Z}+\frac{1}{2}} e^{z(k-\frac{l}{2})}E_{k-l,k} + \frac{\delta_{l,0}}{\varsigma(z)}.$$

These operators generalize the classical bosons. Indeed, for $l \neq 0$,

$$\mathcal{E}_l(0) = \sum_{k\in\mathbb{Z}+\frac{1}{2}} E_{k-l,k} = \alpha_l.$$

Note that $\mathcal{E}_l(z)^* = \mathcal{E}_{-l}(z)$. Similar to the bosons,

$$[\mathscr{H}, \mathcal{E}_{-l}(z)] = l\mathcal{E}_{-l}(z),$$

The operators $\mathcal{E}$ satisfies the commutation relation

$$[\mathcal{E}_l(z), \mathcal{E}_r(w)] = \varsigma\left(\det\begin{bmatrix} l & z \\ r & w \end{bmatrix}\right)\mathcal{E}_{l+r}(z+w). \tag{2.11}$$

In particular,

$$[\alpha_l, \mathcal{E}_r(w)] = \varsigma(lw)\mathcal{E}_{l+r}(w). \tag{2.12}$$

If $l + r \neq 0$, then $\mathcal{E}_{l+r}(w)$ is regular and (2.12) vanishes when $w = 0$. Otherwise the singuler term of $\mathcal{E}_0(w)$ contributes a constant factor as $w \to 0$

$$\frac{\varsigma(lw)}{\varsigma(w)} = \frac{e^{lw/2} - e^{-lw/2}}{e^{w/2} - e^{-w/2}} \xrightarrow{w\to 0} l.$$

This recovers the commutation relation for the bosons

$$[\alpha_l, \alpha_r] = l\delta_{l+r}.$$

By treating z as a formal symbol, we can rewrite $\mathcal{E}_0(z) = \sum_{k\in\mathbb{Z}+\frac{1}{2}} e^{zk}\psi_k\psi_k^*$. From this expression it's easy to see that

$$\mathcal{E}_0(z)\,|\lambda\rangle = \mathbf{e}(\lambda, z)\,|\lambda\rangle\,. \tag{2.13}$$

By extracting the z-coefficients of $\mathcal{E}_0(z)$ we get the operators $\mathcal{P}_k$ for $k > 0$

$$\mathcal{P}_k = k![z^k]\mathcal{E}_0(z), \tag{2.14}$$

then $\mathcal{P}_k |\lambda\rangle = \mathbf{p}_k(\lambda) |\lambda\rangle$ by the definition of $\mathbf{p}_k$. The operator

$$\mathcal{F}_2 = \frac{\mathcal{P}_2}{2!}$$

will play a special role in deriving the special GW/H correspondence.

3. The Gromov-Witten/Hurwitz correspondence

The first three subsections in this section are devoted to the proof of the special GW/H correspondence (1.9). In Section 3.1 the equivariant $n+m$-point functions over $\mathbb{P}^1$ and their generating function are introduced, which will be expressed in some vacuum expectation in Section 3.3, then the special correspondence is captured via the nonequivariant limit of this operator formula.

The operator formula is obtained by two steps: In Section 3.1 the Hodge integrals are introduced in both connected and disconnected fashions, then the generating function for the $n+m$-point functions is expressed in terms of the Hodge integrals using the localization formula — the proof is all about manipulations on the bipartite graghs and the switch between the connected and the disconnected theories. Next, in Section 3.2, we'll show that the generating function for the Hodge interals is an operator formula as a result of the ELSV formula [1] and the Murnaghan-Nakayama rule (2.5). Therefore the $n+m$-point functions and the operator formalism can be bridged by the Hodge integrals through the above two steps.

The last part, Section 3.4, will deal with the full GW/H correspondence:

Theorem 3.1. *For any smooth projective curve X, we have*

$$\left\langle \prod_{i=1}^{n} \tau_{k_i}(\omega), \eta^1, \ldots, \eta^m \right\rangle_d^{\bullet X} = \frac{1}{\prod k_i!} \mathsf{H}_d^X(\overline{(k_1+1)}, \ldots, \overline{(k_n+1)}, \eta^1, \ldots, \eta^m).$$

As a conclusion of the degeneration formula the left hand side will be written in the algebra of shifted symmetric functions. The expression is the same as the right hand side except that it's in some basis which we don't know whether it's the same as the one generated by the completed cycles. The identification of the two bases will finally be confirmed by the special correspondence.

3.1. **Hodge integrals and equivariant $n+m$-point functions.**

This subsection would introduce the formula (3.2), which is an alternating expression of the localization formula in terms of the Hodge integrals.

3.1.1. *Hodge integrals.* Let $\pi : \mathcal{U}_{g,n} \to \overline{M}_{g,n}$ be the universal family and ω_π be the relative dualizing sheaf. Denote by $\Lambda_{g,n}$ the Hodge bundle $\pi_*(\omega_\pi)$ on $\overline{M}_{g,n}$ with total Chern class $c(\Lambda_{g,n})$. The Hodge integral is defined by

$$\mathbb{H}_g^\circ(z_1, \dots, z_n) = \prod_{i=1}^n z_i \int_{\overline{M}_{g,n}} \frac{c(\Lambda_{g,n}^\vee)}{\prod_{i=1}^n (1 - z_i \psi_i)}.$$

The definition can be extended to all non-negative genera and n by directly setting the unstable ones to be some rational functions in those z_i's. For example all the 0-point functions $\mathbb{H}_g^\circ()$ are set to be zero. See [12] for more details.

The full n-point function of the Hodge integrals is defined to be the following generating function over the genera

$$\mathbb{H}^\circ(z, u) = \sum_{g \in \mathbb{Z}} u^{2g-2} \mathbb{H}_g^\circ(z).$$

The corresponding terms $\mathbb{H}_g(z_1, \dots, z_n)$ and $\mathbb{H}(z, u)$ in the disconnected theory can be defined in totally the same way.

3.1.2. *Equivariant $n + m$-point functions.* Let $z = (z_1 \dots, z_n)$ and $w = (w_1 \dots, w_m)$. Consider the $\mathbb{C}^*$ action on $\mathbb{P}^1$ by $t[x : y] = [tx : y]$. The equivariant $n + m$-point function is defined (in disconnected notions) by

$$\mathsf{G}_{g,d}(z, w) = \prod_{i=1}^n z_i \prod_{j=i}^m w_j \int_{[\overline{M}^\bullet_{g,n+m}(\mathbb{P}^1,d)]^{vir}} \prod_{i=1}^n \frac{\mathrm{ev}_i^*(0)}{1 - z_i \psi_i} \prod_{j=i}^m \frac{\mathrm{ev}_j^*(\infty)}{1 - w_j \psi_j}.$$

Similar to the Hodge integral case, the full $n + m$-point function is defined to be

$$\mathsf{G}_d(z, w, u) = \sum_{g \geq 0} u^{2g-2} \mathsf{G}_{g,d}(z, w).$$

Note that by taking $m = 0$, $u = 1$ and the non-equivariant limit $t = 0$, we have

$$\mathsf{G}_d(z, \emptyset, 1)|_{t=0} = \prod_{i=1}^n z_i \int_{\overline{M}^\bullet_{g,n}(\mathbb{P}^1,d)} \prod_{i=1}^n \frac{\mathrm{ev}_i^*(\omega)}{1 - z_i \psi_i} \tag{3.1}$$
$$= \sum_{k_i \geq -1} \left\langle \prod_{i=1}^n \tau_{k_i}(\omega) \right\rangle_d^{\bullet \mathbb{P}^1} \prod z_i^{k_i+1},$$

which is just the generating function for the absolute GW-invariants.

3.1.3. *Localization formula.* The following formula can be obtained by the virtual localization technique [2].

Proposition 3.2. *(Localization formula) For $d \geq 0$,*

$$\mathsf{G}_d(z,w,u) = \sum_{|\mu|=d} \frac{1}{\mathfrak{z}(\mu)} \frac{(u/t)^{l(\mu)}(-u/t)^{l(\mu)}}{t^{d+n}(-t)^{d+m}} \left(\prod \frac{\mu_i^{\mu_i}}{\mu_i!}\right)^2 \mathbb{H}(\mu, tz, \frac{u}{t})\mathbb{H}(\mu, -tw, -\frac{u}{t}). \tag{3.2}$$

Here $\mathfrak{z}(\mu) = |\mathrm{Aut}(\mu)| \prod_{i=1}^{l(\mu)} \mu_i$ *with* $\mathrm{Aut}(\mu) = \prod_{i\geq 1} S_{m_i(\mu)}$ *being the permutation of equal parts of* μ.

Proof. Recall that each fixed component in $\overline{M}^{\bullet}_{g,n+m}(\mathbb{P}^1, d)$ corresponds to a *bipartite graph* Γ consisting of the following data:

- $V = V_0 \coprod V_\infty$: the vertices, which marks contracted components, ramification points or special points mapped to 0 or ∞, respectively. Each $v \in V$ has an assigned genus $g(v)$.
- L_0 = {1,...,n}, L_∞ = {1,...,m}: the legs attached to the vertices, each element stands for a marked point in the vertex mapped to 0, ∞, respectively. The symbol $I(v) \subseteq L_0/L_\infty$ is used to record the markings in the vertex v.
- E: the edge set, which marks the irreducible components dominating the target $\mathbb{P}^1$. Each $e \in E$ has an assigned degree d_e. Around each vertex v, the the set of the incident edges is denoted by $E(v)$ and $e(v) := |E(v)|$. Note that the valence $val(v) = e(v) + |I(v)|$.

Note that Γ may be disconnected since we're in the disconnected theory.

Let $\overline{M}^{\bullet}_{\Gamma} = \prod_{v\in V} \overline{M}_{g(v),val(v)}$, then the corresponding fixed component is an image of the finite cover $\iota : \overline{M}^{\bullet}_{\Gamma} \to \overline{M}^{\bullet}_{g,n+m}(\mathbb{P}^1, d)$, which has Galois group $\mathbb{A}_\Gamma$ consisting of the symmetry of the graph and the automorphism of each dominant irreducible component

$$1 \to \prod_{e\in E} \mathbb{Z}_{d_e} \to \mathbb{A}_\Gamma \to \mathrm{Aut}(\Gamma) \to 1.$$

Now the localization formula can be written down as

$$\mathsf{G}_{g,d}(z,w) = \sum_{\Gamma} \frac{1}{|\mathbb{A}_\Gamma|} \int_{\overline{M}^{\bullet}_{\Gamma}} \frac{1}{e(N_\Gamma^{vir})} \iota * \left(\prod_{i=1}^{n} \frac{\mathrm{ev}_i^*(0)}{1 - z_i\psi_i} \prod_{j=i}^{m} \frac{\mathrm{ev}_j^*(\infty)}{1 - w_j\psi_j} \right).$$

After explicitly evaluateing out the term $e(N_\Gamma^{vir})$, a summand corresponding to the graph Γ can be written as a product where the factors one-to-one correspond to the vertices:

$$\mathsf{G}_{g,d}(z,w) = \sum_{\Gamma} \frac{1}{\prod_{e\in E} d_e} \frac{1}{|\mathrm{Aut}(\Gamma)|} \prod_{v_0\in V_0} C(v_0) \prod_{v_\infty\in V_\infty} C(v_\infty).$$

Those *vertex contributions*, say, for $v_0 \in V_0$, are given by

$$C(v_0) = \int_{\overline{M}_{g(v_0),val(v_0)}} t^{g(v_0)-1} \left(\sum_{i=0}^{g(v_0)} (-1)^i \frac{\lambda_i}{t^i} \right) \prod_{i=1}^{e(v_0)} \left(\frac{d_i^{d_i} t^{-d_i}}{d_i!} \frac{td_i}{t - d_i \psi_i} \right) \prod_{i \in I(v_0)} \frac{tz_i}{1 - z_i \psi_i},$$

where λ_i is the i-th Chern class $c_i(\Lambda_{g,n})$ of the Hodge bundle. The value wouldn't changed by the replacements $\psi \mapsto t\psi$, $\lambda \mapsto t\lambda$ then dividing by $t^{\dim \overline{M}_{g(v_0),val(v_0)}} = t^{3g(v_0)-3+val(v_0)}$, after so the Hodge integral appears

$$C(v_0) = \frac{\prod_{i=1}^{e(v_0)} d_i^{d_i}/d_i!}{t^{2g(v_0)-2+d(v_0)+val(v_0)}} \mathbb{H}^\circ_{g(v_0)}(d_{E(v_0)}, tz_{I(v_0)}),$$

where $d(v) = \sum_{i=1}^{e(v)} d_i$.

$C(v_\infty)$ is obtained by replacing z_i by w_i and t by $-t$ in $C(v_0)$.

To specify the dependence on the genus, the symbol Γ_g will be used in the following. Now consider the generating function

$$\begin{aligned}
\mathsf{G}_d(z, w, u) &= \sum_{g \in \mathbb{Z}} u^{2g-2} \mathsf{G}_{g,d}(z, w) \\
&= \sum_{g \in \mathbb{Z}} u^{2g-2} \sum_{\Gamma_g} \frac{1}{\prod_{e \in E} d_e} \frac{1}{|\mathrm{Aut}(\Gamma_g)|} \prod_{v_0 \in V_0} C(v_0) \prod_{v_\infty \in V_\infty} C(v_\infty) \\
&= \sum_{g \in \mathbb{Z}} u^{2g-2} \sum_{\Gamma_g} \frac{1}{\prod_{e \in E} d_e} \frac{1}{|\mathrm{Aut}(\Gamma_g)|} \\
&\quad \times \prod_{v_0 \in V_0} \frac{\prod_{i=1}^{e(v_0)} d_i^{d_i}/d_i!}{t^{2g(v_0)-2+d(v_0)+val(v_0)}} \mathbb{H}^\circ_{g(v_0)}(d_{E(v_0)}, tz_{I(v_0)}) \\
&\quad \times \prod_{v_\infty \in V_\infty} \frac{\prod_{i=1}^{e(v_\infty)} d_i^{d_i}/d_i!}{(-t)^{2g(v_\infty)-2+d(v_\infty)+val(v_\infty)}} \mathbb{H}^\circ_{g(v_\infty)}(d_{E(v_\infty)}, -tw_{I(v_\infty)}).
\end{aligned}$$

A graph relating to $\overline{M}^\bullet_{g,n+m}(\mathbb{P}^1, d)$ satisfies the following three conditions

- genus: $\sum_{v \in V} (2g(v) - 2 + e(v)) = 2g - 2$,
- degree: $\sum_{v \in V} d(v) = 2d$,
- marking: $\bigcup_{v_0 \in V_0} I(v_0) = \{1, \ldots, n\}$, $\bigcup_{v_\infty \in V_\infty} I(v_\infty) = \{1, \ldots, m\}$.

These conditions help us collecting the terms in the generating function. Note that the assinged degrees on the edges actually form a partition of degree d, therefore we can first sum over graphs Γ^μ with fixed associated partition μ. The following is the rearranged generating function under the above three conditions

$$\mathsf{G}_d(z,w,u)=\sum_{|\mu|=d}\frac{1}{\prod_{i=1}^{l(\mu)}\mu_i}\left(\prod_{i=1}^{l(\mu)}\frac{\mu_i^{\mu_i}}{\mu_i!}\right)^2\left[\frac{(u/t)^{l(\mu)}(-u/t)^{l(\mu)}}{t^{d+n}(-t)^{d+m}}\right]\times$$
$$\sum_{g_0,g_\infty\in\mathbb{Z}}\sum_{\Gamma^\mu_{g_0,g_\infty}}\frac{1}{|\mathrm{Aut}(\Gamma^\mu_{g_0,g_\infty})|}\left(\frac{u}{t}\right)^{2g_0-2}\prod_{v_0\in V_0}\mathbb{H}^\circ_{g(v_0)}(\mu_{E(v_0)},tz_{I(v_0)})$$
$$\times\left(-\frac{u}{t}\right)^{2g_\infty-2}\prod_{v_\infty\in V_\infty}\mathbb{H}^\circ_{g(v_\infty)}(\mu_{E(v_\infty)},-tw_{I(v_\infty)}),$$

where $2g_0-2=\sum_{v_0\in V_0}[2g(v_0)-2]$ and $2g_\infty-2=\sum_{v_\infty\in V_\infty}[2g(v_\infty)-2]$.

From the last formula, we can see that the proof is done if the following equality holds (with $x=\frac{u}{t}$ and $y=-\frac{u}{t}$) for arbitrary variables x and y

$$\mathbb{H}(\mu,tz,x)\mathbb{H}(\mu,-tw,y)\overset{\text{def}}{=}\sum_{g_0,g_\infty\in\mathbb{Z}}x^{2g_0-2}\mathbb{H}_{g_0}(\mu,tz)y^{2g_\infty-2}\mathbb{H}_{g_\infty}(\mu,-tw)$$
$$=\sum_{g_0,g_\infty\in\mathbb{Z}}\sum_{\Gamma^\mu_{g_0,g_\infty}}\frac{|\mathrm{Aut}(\mu)|}{|\mathrm{Aut}(\Gamma^\mu_{g_0,g_\infty})|}x^{2g_0-2}\prod_{v_0\in V_0}\mathbb{H}^\circ_{g(v_0)}(\mu_{E(v_0)},tz_{I(v_0)})$$
$$\times y^{2g_\infty-2}\prod_{v_\infty\in V_\infty}\mathbb{H}^\circ_{g(v_\infty)}(\mu_{E(v_\infty)},-tw_{I(v_\infty)}),$$

or equivalently

$$\mathbb{H}_{g_0}(\mu,tz)\mathbb{H}_{g_\infty}(\mu,-tw)$$
$$=\sum_{\Gamma^\mu_{g_0,g_\infty}}\frac{|\mathrm{Aut}(\mu)|}{|\mathrm{Aut}(\Gamma^\mu_{g_0,g_\infty})|}\prod_{v_0\in V_0}\mathbb{H}^\circ_{g(v_0)}(\mu_{E(v_0)},tz_{I(v_0)})\prod_{v_\infty\in V_\infty}\mathbb{H}^\circ_{g(v_\infty)}(\mu_{E(v_\infty)},-tw_{I(v_\infty)}).$$

By the definition of the Hodge integrals, the problem about the integrations can be turned into the problem among the isomorphicity of the following natural map

$$\coprod_{\Gamma^\mu_{g_0,g_\infty}}\left(\prod_{v_0\in V_0}\overline{M}_{g(v_0),val(v_0)}\times\prod_{v_\infty\in V_0}\overline{M}_{g(v_\infty),val(v_\infty)}\right)\Big/\frac{\mathrm{Aut}(\Gamma^\mu_{g_0,g_\infty})}{\mathrm{Aut}(\mu)}$$
$$\longrightarrow\quad\overline{M}^\bullet_{g_0,l(\mu)+n}\times\overline{M}^\bullet_{g_\infty,l(\mu)+m},$$

where the quotient of $\mathrm{Aut}(\Gamma^\mu_{g_0,g_\infty})$ by $\mathrm{Aut}(\mu)$ means we ignore any permutation among the edges. In fact, for each curve parametrized by $\overline{M}^\bullet_{g_0,l(\mu)+n}\times\overline{M}^\bullet_{g_\infty,l(\mu)+m}$ we can associate uniquely a bipartite graph by, bipartitely, assigning a vertex to each connected component and giving an edge connecting two vertices if they have commom marked points indexed by the components of the partition μ. Actually the unique preimage under the graph is already found in the above decomposition into connected components, so that the map is indeed an isomorphism. □

3.2. **The operator formula for Hodge integrals.** The operator formula for Hodge integrals pointed out a weak connection between the Gromov-Witten theory and the operator formalism. Such connection is built upon the ELSV formula.

Through the extended definition of Hurwitz numbers and the Murnaghan-Nakayama rule (2.5), it will be showed in this subsection that the ELSV formula is actully an operator formula of the Hodge integrals in positive integer variables.

3.2.1. *The ELSV formula.* Given a partition μ of degree d and length l, the Hurwitz number $\mathsf{H}^{\mathbb{P}^1}_{g,d}(\mu,(2)^{\times b})$ enumerates the genus g covers having profile data μ over $\infty \in \mathbb{P}^1$ and simple ramifications over $b = 2g + |\mu| + l(\mu) - 2$ distinct points on $\mathbb{P}^1\backslash\{\infty\}$.

The ELSV formula in the disconnected theory is

$$\mathsf{H}^{\mathbb{P}^1}_{g,d}(\mu,(2)^{\times b}) = \frac{b!}{\mathfrak{z}(\mu)}\prod_{i=1}^{l}\frac{\mu_i^{\mu_i}}{\mu_i!}\mathbb{H}(\mu_1,\ldots,\mu_l).$$

On the other hand, by applying the Murnaghan-Nakayama rule (2.5), we can obtain

$$\begin{aligned}
\mathsf{H}^{\mathbb{P}^1}_{g,d}(\mu,(2)^{\times b}) &= \sum_{|\lambda|=d}\left(\frac{\dim\lambda}{d!}\right)^2 \mathbf{f}_\mu(\lambda)\mathbf{f}_{(2)}(\lambda)^b = \sum_{|\lambda|=d}\left(\frac{\dim\lambda}{d!}\right)^2\left(\frac{|\mu|!\chi^\lambda_\mu}{\mathfrak{z}(\mu)\dim\lambda}\right)\frac{\mathbf{p}_2(\lambda)}{2!}^b \\
&= \sum_{|\lambda|=d}\frac{\dim\lambda}{\mathfrak{z}(\mu)d!}\langle\lambda|\prod_{i=1}^{l}\alpha_{-\mu_i}|0\rangle\langle\lambda|\mathcal{F}_2^b|\lambda\rangle = \sum_{|\lambda|=|\eta|=d}\frac{\dim\lambda}{\mathfrak{z}(\mu)d!}\langle\lambda|\mathcal{F}_2^b|\eta\rangle\langle\eta|\prod_{i=1}^{l}\alpha_{-\mu_i}|0\rangle \\
&= \frac{1}{\mathfrak{z}(\mu)d!}\left(\sum_{|\lambda|=d}\langle\lambda|\dim\lambda\right)\mathcal{F}_2^b\prod_{i=1}^{l}\alpha_{-\mu_i}|0\rangle = \frac{1}{\mathfrak{z}(\mu)d!}\langle 0|\alpha_1^d\mathcal{F}_2^b\prod_{i=1}^{l}\alpha_{-\mu_i}|0\rangle \\
&= \frac{1}{\mathfrak{z}(\mu)}\left\langle e^{\alpha_1}\mathcal{F}_2^b\prod_{i=1}^{l}\alpha_{-\mu_i}\right\rangle.
\end{aligned}$$

By equating the above two quantities we can get

$$\mathbb{H}(\mu_1,\ldots,\mu_n,u) = u^{-|\mu|-l(\mu)}\left(\prod_{i=1}^{n}\frac{\mu_i!}{\mu_i^{\mu_i}}\right)\left\langle e^{\alpha_1}e^{u\mathcal{F}_2}\prod_{i=1}^{n}\alpha_{-\mu_i}\right\rangle.$$

Since the operators $e^{-\alpha_1}$ and $e^{-u\mathcal{F}_2}$ fix the vacuum state, the last equation can be rewritten as

$$\mathbb{H}(\mu_1, \ldots, \mu_n, u) = u^{-|\mu|-l(\mu)} \left(\prod_{i=1}^{n} \frac{\mu_i!}{\mu_i^{\mu_i}} \right) \left\langle \prod_{i=1}^{n} \left(e^{\alpha_1} e^{u\mathcal{F}_2} \alpha_{-\mu_i} e^{-u\mathcal{F}_2} e^{-\alpha_1} \right) \right\rangle. \tag{3.3}$$

3.2.2. *The operators $\mathcal{A}$.* The operators $\mathcal{A}$ is defined by

$$\mathcal{A}(a, b) = \mathcal{S}(b)^a \sum_{k \in \mathbb{Z}} \frac{\varsigma(b)^k}{(a+1)_k} \mathcal{E}_k(b),$$

in which the following standard notation is used

$$(a+1)_k = \frac{(a+k)!}{a!} = \begin{cases} (a+1)(a+2)\cdots(a+k) & k \geq 0 \\ \frac{1}{a(a-1)\cdots(a+k+1)} & k \leq 0. \end{cases}$$

Lemma 3.3.

$$e^{\alpha_1} e^{u\mathcal{F}_2} \alpha_{-m} e^{-u\mathcal{F}_2} e^{-\alpha_1} = \frac{u^m m^m}{m!} \mathcal{A}(m, um), \quad \forall m \in \mathbb{N}.$$

Proof. By (2.12) we have $[\alpha_1, \mathcal{E}_{-m}(s)] = \varsigma(s)\mathcal{E}_{-m+1}(s)$, thus by the useful formula (2.4) we can get

$$e^{\alpha_1} \mathcal{E}_{-m}(s) e^{-\alpha_1} = \frac{\varsigma(s)^m}{m!} \sum_{k \in \mathbb{Z}} \frac{\varsigma(s)^k}{(m+1)_k} \mathcal{E}_k(s). \tag{3.4}$$

Again by (2.12) and (2.4) we can get that

$$e^{u\mathcal{F}_2} \alpha_{-m} e^{-u\mathcal{F}_2} = \mathcal{E}_{-m}(um).$$

Now the proof is done by placing the above formula between e^{α_1} and $e^{-\alpha_1}$ and then using (3.4). □

Combine (3.3) with the above lemma we can get

$$\mathbb{H}(\mu_1, \ldots, \mu_n, u) = u^{-n} \left\langle \prod_{i=1}^{n} \mathcal{A}(\mu_i, u\mu_i) \right\rangle.$$

The above formula says that the following formula would hold when $z_1, \ldots, z_n \in \mathbb{N}$

$$\mathbb{H}(z_1, \ldots, z_n, u) = u^{-n} \left\langle \prod_{i=1}^{n} \mathcal{A}(z_i, uz_i) \right\rangle. \tag{3.5}$$

The identity can be proved true further in formal variables $z_1, \ldots, z_n$ through a rather lengthy discussion on the analytic property, which is omitted in this article and is refered to [12].

3.3. **Special GW/H correspondence.** For every partition μ of fixed degree d, define

$$\left|\chi_\mu\right\rangle = \sum_{|\lambda|=|\mu|} \chi_\mu^\lambda \left|\lambda\right\rangle \overset{(2.5)}{=} \sum_{i=1}^{l(\mu)} \alpha_{-\mu_i} \left|0\right\rangle .$$

These states form an orthogonal system in the zero-charge, energy-d subspace $\mathfrak{F}_0^d$, indeed

$$\left\langle\chi_\mu\middle|\chi_\nu\right\rangle = \left\langle \prod_{j=1}^{l(\nu)} \alpha_{\nu_j} \prod_{i=1}^{l(\mu)} \alpha_{-\mu_i} \right\rangle = \mathfrak{z}(\mu)\delta_{\mu,\nu},$$

where the last equality follows from the commutation relation (2.1). By comparing the dimension and the amount of such states, we found that $\left\{\left|\chi_\mu\right\rangle\right\}_{|\mu|=d}$ actually forms an orthogonal basis for $\mathfrak{F}_0^d$.

3.3.1. *Projection operators.* Define $P_\varnothing = \left|0\right\rangle\left\langle 0\right|$ to be the orthogonal projection onto the 1-dimensional zero-energy subspace $\mathfrak{F}^0 = \mathfrak{F}_0^0 = \mathbb{C}\left|0\right\rangle$. In addition we define the orthogonal projection onto the energy-d subspace $\mathfrak{F}^d$ for each $d > 0$ to be

$$P_d = \sum_{|\mu|=d} \frac{1}{\mathfrak{z}(\mu)} \left|\chi_\mu\right\rangle\left\langle\chi_\mu\right|$$

For any operators X and Y we have

$$\left\langle X\right\rangle\left\langle Y\right\rangle = \left\langle XP_\varnothing Y^*\right\rangle , \tag{3.6}$$

which indicates that P_d can be obtained from $P_\varnothing$ as in the following

$$P_d = \sum_{|\mu|=d} \frac{1}{\mathfrak{z}(\mu)} \prod_{i=1}^{l(\mu)} \alpha_{-\mu_i} P_\varnothing \prod_{i=1}^{l(\mu)} \alpha_{\mu_i}. \tag{3.7}$$

3.3.2. *Special GW/H correspondence.* Now we're ready to derive the special GW/H correspondence. As a first step, we want to rewrite the localization formula (3.2) in some vacuum expectations, and this requires (3.5):

$$\begin{aligned}
&\frac{(u/t)^{l(\mu)}}{t^{d+n}} \left(\prod \frac{\mu_i^{\mu_i}}{\mu_i!}\right)^2 \mathbb{H}(\mu, tz, \frac{u}{t}) \\
&= t^{-d}u^{-n} \left(\prod \frac{\mu_i^{\mu_i}}{\mu_i!}\right) \left\langle \prod_{i=1}^{n} \mathcal{A}(tz_i, uz_i) \prod_{i=1}^{l(\mu)} \mathcal{A}(\mu_i, \frac{u}{t}\mu_i) \right\rangle \\
&= u^{-d-n} \left\langle \prod_{i=1}^{n} \mathcal{A}(tz_i, uz_i) e^{\alpha_1} e^{\frac{u}{t}\mathcal{F}_2} \prod_{i=1}^{l(\mu)} \alpha_{-\mu_i} \right\rangle \quad \text{(by Lemma 3.3).}
\end{aligned}$$

Similarly,

$$\frac{(-u/t)^{l(\mu)}}{-t^{d+n}}\left(\prod\frac{\mu_i^{\mu_i}}{\mu_i!}\right)^2 \mathbb{H}(\mu,-tw,-\frac{u}{t})$$
$$= u^{-d-m}\left\langle\prod_{j=1}^{m}\mathcal{A}(-tw_j,uw_j)e^{\alpha_1}e^{-\frac{u}{t}\mathcal{F}_2}\prod_{i=1}^{l(\mu)}\alpha_{-\mu_i}\right\rangle.$$

Together with (3.6) and (3.7), we get

$$u^{2d+n+m}\mathsf{G}_d(z,w,u)$$
$$= \sum_{|\mu|=d}\frac{1}{\mathfrak{z}(\mu)}\left\langle\prod_{i=1}^{n}\mathcal{A}(tz_i,uz_i)e^{\alpha_1}e^{\frac{u}{t}\mathcal{F}_2}\prod_{i=1}^{l(\mu)}\alpha_{-\mu_i}\right\rangle\left\langle\prod_{j=1}^{m}\mathcal{A}(-tw_j,uw_j)e^{\alpha_1}e^{-\frac{u}{t}\mathcal{F}_2}\prod_{i=1}^{l(\mu)}\alpha_{-\mu_i}\right\rangle$$
$$= \sum_{|\mu|=d}\frac{1}{\mathfrak{z}(\mu)}\left\langle\prod_{i=1}^{n}\mathcal{A}(tz_i,uz_i)e^{\alpha_1}e^{\frac{u}{t}\mathcal{F}_2}\prod_{i=1}^{l(\mu)}\alpha_{-\mu_i}P_\varnothing\prod_{i=1}^{l(\mu)}\alpha_{\mu_i}e^{-\frac{u}{t}\mathcal{F}_2}e^{\alpha_{-1}}\prod_{j=1}^{m}\mathcal{A}(-tw_j,uw_j)^*\right\rangle$$
$$= \left\langle\prod_{i=1}^{n}\mathcal{A}(tz_i,uz_i)e^{\alpha_1}e^{\frac{u}{t}\mathcal{F}_2}P_de^{-\frac{u}{t}\mathcal{F}_2}e^{\alpha_{-1}}\prod_{j=1}^{m}\mathcal{A}(-tw_j,uw_j)^*\right\rangle$$
$$= \left\langle\prod_{i=1}^{n}\mathcal{A}(tz_i,uz_i)e^{\alpha_1}P_de^{\alpha_{-1}}\prod_{j=1}^{m}\mathcal{A}(-tw_j,uw_j)^*\right\rangle.$$

Note that the last equality holds because $[\mathscr{H},\mathcal{F}_2]=0$ implies $[P_d,\mathcal{F}_2]=0$.

We've proved the following theorem

Theorem 3.4.

$$\mathsf{G}_d(z,w,u) = u^{-2d-n-m}\left\langle\prod_{i=1}^{n}\mathcal{A}(tz_i,uz_i)e^{\alpha_1}P_de^{\alpha_{-1}}\prod_{j=1}^{m}\mathcal{A}(-tw_j,uw_j)^*\right\rangle.$$

From (3.4):

$$\mathcal{A}(0,z)\overset{def}{=}\sum_{k\geq 0}\frac{\varsigma(z)^k}{k!}\mathcal{E}_k(z)=e^{\alpha_1}\mathcal{E}_0(z)e^{-\alpha_1}.$$

Hence by taking $m=0$, $u=1$, and $t=0$, Theorem 3.4 reduces to the following form

$$\mathsf{G}_d(z,\varnothing,1)|_{t=0}=\left\langle\prod\mathcal{A}(o,z_i)e^{\alpha_1}P_de^{\alpha_{-1}}\right\rangle=\left\langle e^{\alpha_1}\prod\mathcal{E}_0(z_i)P_de^{\alpha_{-1}}\right\rangle.$$

In the last term, because only energy-d states survive under P_d, we can replace the operators e^{α_1} and $e^{\alpha_{-1}}$ by α_1^d and α_{-1}^d respectively, and this leads to the special GW/H correspondence in the operator language

$$\mathsf{G}_d(z,\varnothing,1)|_{t=0}=\frac{1}{(d!)^2}\left\langle\alpha_1^d\prod\mathcal{E}_0(z_i)\alpha_{-1}^d\right\rangle. \tag{3.8}$$

Corollary. *(Special GW/H correspondence)*

$$\left\langle \prod_{i=1}^{n} \tau_{k_i}(\omega) \right\rangle_d^{\bullet \mathbb{P}^1} = \frac{1}{\prod k_i!} \mathsf{H}_d^{\mathbb{P}^1}(\overline{(k_1+1)}, \ldots, \overline{(k_n+1)}).$$

Proof. Murnaghan-Nakayama rule (2.5) in $\nu = 1^d$ would be

$$\alpha_{-1}^d |0\rangle = \sum_{|\lambda|=d} \chi_{1^d}^{\lambda} |\lambda\rangle = \sum_{|\lambda|=d} (\dim \lambda) |\lambda\rangle .$$

Together with (2.13), the right hand side of (3.8) can now be rewritten as:

$$\begin{aligned} RHS &= \sum_{|\lambda|=|\eta|=d} \frac{1}{(d!)^2} (\dim \lambda)(\dim \eta) \langle \eta | \prod \mathcal{E}_0(z_i) |\lambda\rangle \\ &= \sum_{|\lambda|=|\eta|=d} \frac{1}{(d!)^2} (\dim \lambda)(\dim \eta) \prod \mathbf{e}(\lambda, z_i) \langle \eta | \lambda \rangle \\ &= \sum_{|\lambda|=d} \left(\frac{\dim \lambda}{d!} \right)^2 \prod \mathbf{e}(\lambda, z_i), \end{aligned}$$

where the coefficient in $\prod z_i^{k_i+1}$ is

$$\frac{1}{\prod k_i!} \sum_{|\lambda|=d} \left(\frac{\dim \lambda}{d!} \right)^2 \prod_{i=1}^{n} \frac{\mathbf{p}_{k_i+1}(\lambda)}{k_i+1} = \frac{1}{\prod k_i!} \mathsf{H}_d^{\mathbb{P}^1}(\overline{(k_1+1)}, \ldots, \overline{(k_n+1)}).$$

The proof is done after comparing these coefficients with those appeared in (3.1). □

3.4. **Full GW/H correspondence.** The goal of this section is the full correspondence Theorem 3.1, or more explicitly,

$$\begin{aligned} &\left\langle \prod_{i=1}^{n} \tau_{k_i}(\omega), \eta^1, \ldots, \eta^m \right\rangle_d^{\bullet X} \\ &= \sum_{|\lambda|=d} \left(\frac{\dim \lambda}{d!} \right)^{2-2g(X)} \prod_{j=1}^{m} \mathbf{f}_{\eta^j}(\lambda) \prod_{i=1}^{n} \frac{\mathbf{p}_{k_i+1}(\lambda)}{(k_i+1)!}. \end{aligned}$$

3.4.1. *The degeneration formula.* Let X be a smooth curve with n distinct points $x_1, \ldots, x_n \in X$. We apply degeneration to the normal cone to the pair $(X, x_1, \ldots, x_n)$ to get the family

$$\mathfrak{X} \to \mathbb{A}^1,$$

then it follows immediately from the degeneration formula that

$$\begin{aligned}(3.9)\qquad &\left\langle \prod_{i=1}^{n} \tau_{k_i}(\omega), \eta^1, \ldots, \eta^m \right\rangle_d^{\bullet X} \\ &= \sum_{|\mu^i|=d} \mathsf{H}_d^X(\mu^1, \ldots, \mu^n, \eta^1, \ldots, \eta^m) \prod_{i=1}^{n} \mathfrak{z}(\mu^i) \left\langle \mu^i, \tau_{k_i}(\omega) \right\rangle^{\bullet \mathbb{P}^1} \\ &= \sum_{|\lambda|=d} \left(\frac{\dim \lambda}{d!} \right)^{2-2g(X)} \prod_{j=1}^{m} \mathbf{f}_{\eta^j}(\lambda) \prod_{i=1}^{n} \sum_{|\mu^i|=d} \mathbf{f}_{\mu^i}(\lambda) \mathfrak{z}(\mu^i) \left\langle \mu^i, \tau_{k_i}(\omega) \right\rangle^{\bullet \mathbb{P}^1} .\end{aligned}$$

From this formula we can see that the remaining things to do is to show that actually the sum

$$(3.10)\qquad \sum_{|\mu|=d} \mathbf{f}_\mu(\lambda) \mathfrak{z}(\mu) \left\langle \mu, \tau_k(\omega) \right\rangle^{\bullet \mathbb{P}^1}$$

is exactly the function

$$\frac{\mathbf{p}_{k+1}(\lambda)}{(k+1)!}.$$

Note that

$$\langle \nu \rangle^{\bullet \mathbb{P}^1} = \frac{\delta_{\nu, 1^{|\nu|}}}{|\nu|!}.$$

In other words, the only nontrivial no-insertion relative invariant over one point of $\mathbb{P}^1$ occurs when the profile datum is trivial. This fact allows us to easily decompose the invariants into a sum in the connected theory:

$$\left\langle \mu^i, \tau_{k_i}(\omega) \right\rangle^{\bullet \mathbb{P}^1} = \sum_{i=0}^{m_1(\mu)} \frac{1}{i!} \left\langle \mu - 1^i, \tau_k(\omega) \right\rangle^{\circ \mathbb{P}^1} .$$

Since

$$\frac{\mathfrak{z}(\mu)}{i!} = \binom{m_1(\mu)}{i} \mathfrak{z}(\mu - 1^i),$$

we have

$$\mathfrak{z}(\mu) \left\langle \mu, \tau_k(\omega) \right\rangle^{\bullet \mathbb{P}^1} = \sum_{i=0}^{m_1(\mu)} \binom{m_1(\mu)}{i} \mathfrak{z}(\mu - 1^i) \left\langle \mu - 1^i, \tau_k(\omega) \right\rangle^{\circ \mathbb{P}^1} .$$

And because $\mathbf{f}_{\mu - 1^i}(\lambda) = \binom{m_1(\mu)}{i} \mathbf{f}_\mu(\lambda)$, we can rewrite (3.10) as

$$(3.11)\qquad \sum_{\nu} \mathbf{f}_\nu(\lambda) \mathfrak{z}(\nu) \left\langle \nu, \tau_k(\omega) \right\rangle^{\circ \mathbb{P}^1} .$$

Note that now the sum is over all partitions.

3.4.2. *The leading term.* The dimension constraint for $\langle \nu, \tau_k(\omega)\rangle^{\circ\mathbb{P}^1}$ is

$$2g - 1 + |\nu| + l(\nu) = k + 1.$$

Since $g \geq 0$ and $l(\nu) \geq 1$, we have

$$|\nu| \leq k + 1,$$

where equality holds iff $g = 0$ and $\nu = (k + 1)$. Therefore the only leading term of (3.11) occurs with $\nu = (k + 1)$.

Lemma 3.5. *For $d > 0$, we have*

$$\langle (d), \tau_{d-1}(\omega)\rangle^{\circ\mathbb{P}^1} = \frac{1}{d!}.$$

Proof. Only the smooth locus $M_{0,1}(\mathbb{P}^1, (d))$ contributes to the invariant. Indeed, the profile data (d) inhibits the occurrence of more than one irreducible component dominating the target space. Next, if there were a moduli point having its only marked point on a contracted component, then this component must intersect the complementary part once more to fit the stability. But this would create a loop in the domain curve and thus contradicts the $g = 0$ constraint.

Additionally, if we know that the cycle

$$(d-1)!c_1(L_1)^{d-1}\mathrm{ev}_1^*(\omega) \cap [M_{0,1}(\mathbb{P}^1, (d))] \in A_0(M_{0,1}(\mathbb{P}^1, (d)))$$

parametrizes the covers enumerated by the Hurwitz number $\mathsf{H}_d^{\mathbb{P}^1}((d), (d)) = \frac{1}{d}$, then

$$\begin{aligned}\langle (d), \tau_{d-1}(\omega)\rangle^{\circ\mathbb{P}^1} &= \int_{M_{0,1}(\mathbb{P}^1,(d))} c_1(L_1)^{d-1}\mathrm{ev}_1^*(\omega)\\ &= \frac{1}{(d-1)!}\mathsf{H}_d^{\mathbb{P}^1}((d),(d)) = \frac{1}{d!}.\end{aligned}$$

To prove the above assumption, first let $\omega = 0 \in \mathbb{P}^1$ and apply the Chow theory on $\mathrm{ev}_1^{-1}(0)$ instead of $M_{0,1}(\mathbb{P}^1, (d))$.

A natural section of the line bundle L_1 on $\mathrm{ev}_1^{-1}(0)$ can be constructed as follows. For each moduli point $[C, f, p_1] \in \mathrm{ev}_1^{-1}(0)$, consider the map

$$T_0^*\mathbb{P}^1 \cong \frac{\mathfrak{m}_0}{\mathfrak{m}_0^2} \quad \longrightarrow \quad T_{p_1}^*C \cong \frac{\mathfrak{m}_{p_1}}{\mathfrak{m}_{p_1}^2},$$

where $\mathfrak{m}_0$ and $\mathfrak{m}_{p_1}$ are the maximal ideals corresponding to the point $0 \in \mathbb{P}^1$ and $p_1 \in C$ respectively. Fixing the identification $T_0^*\mathbb{P}^1 \cong \mathbb{C}$, then the image of $1 \in T_0^*\mathbb{P}^1$ will give us the required section.

Note that the zero loci of this section stands for the covers having ramification order at p_1 greater than 1. And $c_1(L_1) \cap \mathrm{ev}_1^{-1}(0)$ coincides with the zero loci without concerning the reduceness problem.

Next, consider the Chow theory on $c_1(L_1) \cap \mathrm{ev}_1^{-1}(0)$. Now we have the map

$$T_0^*\mathbb{P}^1 \cong \frac{\mathfrak{m}_0}{\mathfrak{m}_0^2} \quad \longrightarrow \quad T_{p_1}^*C^{\otimes 2} \cong \frac{\mathfrak{m}_{p_1}^2}{\mathfrak{m}_{p_1}^3}.$$

Again a natural section is given by the image of $1 \in T_0^*\mathbb{P}^1$. And set-theoretically the zero loci of the section, $2c_1(L_1) \cap c_1(L_1) \cap \mathrm{ev}_1^{-1}(0)$, represents the covers ramified at p_1 with order greater than 2.

Repeating the above process, we can get that the cycle

$$(d-1)!c_1(L_1)^{d-1} \cap \mathrm{ev}_1^{-1}(0)$$

set-theoretically represents the covers ramified at p_1 with order greater than $d-1$. Actually it's exactly $d-1$ and simple ramified elsewhere by the Riemann-Hurwitz formula. Therefore the cycle supports on the covers enumerated by the number $\mathsf{H}_d^{\mathbb{P}^1}((d),(d))$.

Up to now the only missing piece is the reduceness of each zero loci of the section created in each of the above step.

Here is an elementary way to see it: Suppose the profile data (d) is over $\infty \in \mathbb{P}^1$. Given any moduli point $[C, f] \in \mathrm{ev}_1^{-1}(0)$, let z be the coordinate on $C \cong \mathbb{P}^1$ and w the coordinate on the target space $\mathbb{P}^1$. W.L.O.G. we may set $p_1 = 0$. The map f can then be explicitly written as

$$w = f(z) = a_1 z + a_2 z^2 + \cdots + a_d z^d.$$

Note that the parameters $a_0, \ldots, a_d$ for the stable maps can be seen as the coordinates on $M_{0,1}(\mathbb{P}^1,(d))$.

We have the following induced map

$$\begin{aligned} T_0^*\mathbb{P}^1 &\longrightarrow \Omega_{C,p_1} \\ dw &\longmapsto \left(a_1 + 2a_2 z + \cdots + da_d z^{d-1}\right) dz. \end{aligned}$$

Observe that this map depends linearly on the coordinates $a_i' s$. It follows that the induced section in each level of the above induction process has reduced zero locus. □

3.4.3. *The full GW/H correspondence.* Combining all the previous results, we can define the function $\tilde{\mathbf{p}}_k$ by

$$\begin{aligned} \frac{\tilde{\mathbf{p}}_{k+1}}{(k+1)!} &= \sum_\nu \mathbf{f}_\nu(\lambda)\mathfrak{z}(\nu) \langle \nu, \tau_k(\omega) \rangle^{\circ\mathbb{P}^1} \\ &\overset{(3.5)}{=} \frac{\mathbf{f}_{(k+1)}(\lambda)}{k!} + \text{(lower degree terms)} \\ &\overset{(1.8)}{=} \frac{\mathbf{p}_{k+1}}{(k+1)!} + \text{(lower degree terms)}. \end{aligned}$$

From this, we see that the transformation matrix between the bases $\{\tilde{\mathbf{p}}_\mu\}$ and $\{\mathbf{p}_\mu\}$ is unitriangular, where $\tilde{\mathbf{p}}_\mu = \prod_{i=1}^{l(\mu)} \tilde{\mathbf{p}}_{\mu_i}$.

The final step is to show that the matrix is orthogonal. This can be attained by constructing an inner product on the algebra Λ^*:

For any $f \in \Lambda^*$, define l to be a linear form on Λ^* by

$$l(f) = \sum_\lambda \left(\frac{\dim \lambda}{|\lambda|!} \right)^2 f(\lambda).$$

It's straightforward to check that the series converges. The associated quadratic form is defined by

$$(f, g) \longmapsto l(fg),$$

which is clearly positive definite.

By the *special GW/H correspondence* (1.9),

$$l(\tilde{\mathbf{p}}_k) = l(\mathbf{p}_k) \quad \forall k \in \mathbb{N}.$$

It implies that for any partitions μ, ν,

$$(\tilde{\mathbf{p}}_\mu, \tilde{\mathbf{p}}_\nu) = (\mathbf{p}_\mu, \mathbf{p}_\nu).$$

Therefore the transformation matrix is orthogonal under the associated inner product, and finally the proof for the GW/H correspondence is done.

4. Main results

Consider an operator M on $\mathfrak{F}$ such that

$$[M \otimes M, \Omega] = 0, \quad \Omega = \sum_{k \in \mathbb{Z} + \frac{1}{2}} \psi_k \otimes \psi_k^*. \tag{4.1}$$

The τ-function is defined with respect to the operator M by

$$\tau_n^M(t, s) = \left\langle T^{-n} \Gamma_+(t) M \Gamma_-(s) T^n \right\rangle.$$

For simplicity, the superscript M will be omitted in what follows.

In the first section, it will be showed that the functions $\{\tau_n\}$ satisfies a system of equations, called the Hirota equations for the Toda hierarchy. In particular the lowest degree one is

$$\frac{\partial^2}{\partial t_1 \partial s_1} \log \tau_n = \frac{\tau_{n+1} \tau_{n-1}}{\tau_n^2}, \quad n \in \mathbb{Z}. \tag{4.2}$$

In the second section a τ-function is defined to be a generating function of the relative GW-invariants over two point of $\mathbb{P}^1$. We'll find that it is a special case of the τ-function in the previous paragraph with

$$M = \exp\left(\sum_{k=0}^{\infty} \frac{x_k}{(k+1)!}\mathcal{P}_{k+1}\right), \{x_k\}\text{: a sequence of indeterminants.}$$

Therefore the τ-function here satisfies the Toda hierarchy, in particular the equation (4.2).

4.1. **Hirota equations for the Toda hierarchy.** We define the τ-function to be

$$\tau_n(t,s) = \langle n|\, \hat{M}\, |n\rangle\,, \quad \hat{M} = \Gamma_+(t) M \Gamma_-(s).$$

By (2.9) $\Gamma_\pm(t)$ commutes with $\psi(z)\otimes\psi^*(z)$, so $\hat{M}$ also satisfies (4.1). Note that $\Omega = \sum_{k\in\mathbb{Z}+\frac{1}{2}} \psi_k \otimes \psi_k^* = [z^0]\psi(z)\otimes\psi^*(z)$, hence the commutation relation can be revealed as

$$[z^0]\left(\hat{M}\otimes\hat{M}\right)(\psi(z)\otimes\psi^*(z)) = [z^0](\psi(z)\otimes\psi^*(z))\left(\hat{M}\otimes\hat{M}\right). \tag{4.3}$$

Now we evaluate both sides of (4.3) at the following two states:

$$\Gamma_-(s')\,|n\rangle \otimes \Gamma_-(-s')\,|l+1\rangle\,, \quad \Gamma_-(t')\,|n+1\rangle \otimes \Gamma_-(-t')\,|l\rangle\,.$$

For the left hand side, we have

$$\begin{aligned}
&(\langle n+1|\,\Gamma_+(t') \otimes \langle l|\,\Gamma_+(-t'))\ \text{LHS}\ (\Gamma_-(s')\,|n\rangle \otimes \Gamma_-(-s')\,|l+1\rangle)\\
&= [z^0]\,\underbrace{\langle n+1|\,\Gamma_+(t')\hat{M}\psi(z)\Gamma_-(s')\,|n\rangle}_{(1)}\ \underbrace{\langle l|\,\Gamma_+(-t')\hat{M}\psi^*(z)\Gamma_-(-s')\,|l+1\rangle}_{(2)} \sim (\dagger)
\end{aligned}$$

$$\begin{aligned}
(1) &\overset{(2.9)}{=} \gamma(z^{-1},-s')\,\langle n+1|\,\Gamma_+(t')\hat{M}\Gamma_-(s')\psi(z)\,|n\rangle\\
&\overset{(2.10)}{=} \gamma(z^{-1},-s')\,\langle n+1|\,\Gamma_+(t+t')M\Gamma_-(s+s')z^C T\Gamma_-(z)\Gamma_+(-z^{-1})\,|n\rangle\\
&\overset{(2.6)}{=} \gamma(z^{-1},-s')\,\langle n+1|\,\Gamma_+(t+t')M\Gamma_-(s+s'+z)z^C T\,|n\rangle\\
&= z^{n+1}\gamma(z^{-1},-s')\,\langle n+1|\,\Gamma_+(t+t')M\Gamma_-(s+s'+z)\,|n+1\rangle\\
&= z^{n+1}\gamma(z^{-1},-s')\tau_{n+1}(t+t', s+s'+z)\\
(2) &\overset{(2.9)}{=} \gamma(z^{-1},-s')\,\langle l|\,\Gamma_+(-t')\hat{M}\Gamma_-(-s')\psi^*(z)\,|l+1\rangle\\
&\overset{(2.10)}{=} \gamma(z^{-1},-s')\,\langle l|\,\Gamma_+(t-t')M\Gamma_-(s-s')T^{-1}z^{-C}\Gamma_-(-z)\Gamma_+(z^{-1})\,|l+1\rangle\\
&\overset{(2.6)}{=} \gamma(z^{-1},-s')\,\langle l|\,\Gamma_+(t-t')M\Gamma_-(s-s'-z)T^{-1}z^{-C}\,|l+1\rangle\\
&= z^{-l-1}\gamma(z^{-1},-s')\,\langle l|\,\Gamma_+(t-t')M\Gamma_-(s-s'-z)\,|l\rangle\\
&= z^{-l-1}\gamma(z^{-1},-s')\tau_l(t-t', s-s'-z)\\
&\Rightarrow (\dagger) = [z^{l-n}]\gamma(z^{-1},-2s')\tau_{n+1}(t+t', s+s'+z)\tau_l(t-t', s-s'-z).
\end{aligned}$$

The evaluation for the right hand side is almost the same thus skipped.

The above leads to the following so-called Hirota equations for the Toda hierarchy

$$\begin{aligned}(4.4)\qquad &[z^{l-n}]\gamma(z^{-1},-2s')\tau_{n+1}(t+t',s+s'+z)\tau_l(t-t',s-s'-z)\\ &=[z^{n-l}]\gamma(z^{-1},2s)\tau_n(t+t'-z,s+s')\tau_{l+1}(t-t'+z,s-s').\end{aligned}$$

In particular, let $l=n-1$, Taylor expand both sides with respect to (t,t') and then take out the coefficients of s_1' we can get

$$\begin{aligned}\text{LHS:}\quad &[z^{-1}s_1'](1-2s_1'z^{-1}+\ldots)(\tau_{n+1}+\frac{\partial\tau_{n+1}}{\partial s_1}s_1'+\ldots)(\tau_{n-1}+\frac{\partial\tau_{n-1}}{\partial s_1}s_1')\\ &=-2\tau_{n+1}\tau_{n-1}\\ \text{RHS:}\quad &[z^1s_1'](1+\ldots)(\tau_n-\frac{\partial\tau_n}{\partial t_1}z+\frac{\partial\tau_n}{\partial s_1}s_1'+\frac{\partial^2\tau_n}{\partial t_1\partial s_1}zs_1'+\ldots)\\ &\qquad\times(\tau_n+\frac{\partial\tau_n}{\partial t_1}z-\frac{\partial\tau_n}{\partial s_1}s_1'+\frac{\partial^2\tau_n}{\partial t_1\partial s_1}zs_1'+\ldots)\\ &=-2\tau_n\frac{\partial^2\tau_n}{\partial t_1\partial s_1}+2\frac{\partial\tau_n}{\partial s_1}\frac{\partial\tau_n}{\partial t_1}.\end{aligned}$$

So we have

$$\tau_{n+1}\tau_{n-1}=\tau_n\frac{\partial^2\tau_n}{\partial t_1\partial s_1}-\frac{\partial\tau_n}{\partial s_1}\frac{\partial\tau_n}{\partial t_1},$$

which is exactly (4.2).

4.2. **The Toda hierarchy in the GW-theory.** Consider the following generating function

$$(4.5)\qquad \mathsf{F}^\bullet_{\mu,\nu}(z_1,\ldots,z_n)=\sum_{k_i=-\infty}^{\infty}\left\langle\mu,\prod_{i=1}^n\tau_{k_i}(\omega),\nu\right\rangle^{\bullet\mathbb{P}^1}\prod_{i=1}^n z_i^{k_i+1}.$$

Note that the convention for $\tau_k(\omega)$ as $k<0$ is used:

$$\tau_k(\omega)=\begin{cases}1 & \text{if } k=-2,\\ 0 & \text{if } k\neq-2.\end{cases}$$

The generating function(4.5) can be rewritten as an operator formula

$$(4.6)\qquad \mathsf{F}^\bullet_{\mu,\nu}(z_1,\ldots,z_n)=\frac{1}{\mathfrak{z}(\mu)\mathfrak{z}(\nu)}\left\langle\prod_{i=1}^{l(\mu)}\alpha_{\mu_i}\prod_{i=1}^n\mathcal{E}_0(z_i)\prod_{i=1}^{l(\nu)}\alpha_{-\nu_i}\right\rangle.$$

Indeed, by the GW/H correspondence (3.1)

$$\left\langle \mu, \prod_{i=1}^{n} \tau_{k_i}(\omega), \nu \right\rangle_d^{\bullet\mathbb{P}^1} = \frac{1}{\prod k_i!} \mathsf{H}_d^{\mathbb{P}^1}(\mu, \nu, \overline{(k_1+1)}, \dots, \overline{(k_n+1)})$$

$$= \sum_{|\lambda|=d} \left(\frac{\dim \lambda}{d!}\right)^2 \mathbf{f}_\mu(\lambda)\mathbf{f}_\nu(\lambda) \prod_{i=1}^{n} \frac{\mathbf{p}_{k_i+1}(\lambda)}{(k_i+1)!} \overset{(1.5)}{=} \sum_{|\lambda|=d} \frac{|C_\mu|}{d!}\frac{|C_\nu|}{d!} \chi^\lambda_\mu \chi^\lambda_\nu \prod_{i=1}^{n} \frac{\mathbf{p}_{k_i+1}(\lambda)}{(k_i+1)!}$$

$$\overset{|C_\mu| = \frac{d!}{\mathfrak{z}(\mu)}}{=} \frac{1}{\mathfrak{z}(\mu)\mathfrak{z}(\nu)} \sum_{|\lambda|=d} \chi^\lambda_\mu \chi^\lambda_\nu \prod_{i=1}^{n} \frac{\mathbf{p}_{k_i+1}(\lambda)}{(k_i+1)!}.$$

Here the corresponding convention for $\mathbf{p}_k$ when $k \leq 0$ is

$$\mathbf{p}_0 \equiv 0, \quad \frac{\mathbf{p}_{-1}}{(-1)!} \equiv 1, \quad \text{and } \mathbf{p}_k \equiv 0 \text{ if } k \leq -2.$$

Now the generating function becomes

$$\mathsf{F}^\bullet_{\mu,\nu}(z_1, \dots, z_n) = \frac{1}{\mathfrak{z}(\mu)\mathfrak{z}(\nu)} \sum_{|\lambda|=|\mu|} \chi^\lambda_\mu \chi^\lambda_\nu \prod_{i=1}^{n} \mathbf{e}(\lambda, z_i)$$

$$\overset{(2.13)}{=} \frac{1}{\mathfrak{z}(\mu)\mathfrak{z}(\nu)} \sum_{|\lambda|=|\mu|} \chi^\lambda_\mu \chi^\lambda_\nu \prod_{i=1}^{n} \langle \lambda | \mathcal{E}(z_i) | \lambda \rangle$$

$$= \frac{1}{\mathfrak{z}(\mu)\mathfrak{z}(\nu)} \left(\sum_{|\lambda|=|\mu|} \langle \lambda | \chi^\lambda_\mu \right) \prod_{i=1}^{n} \mathcal{E}(z_i) \left(\sum_{|\lambda|=|\nu|} \chi^\lambda_\nu | \lambda \rangle \right).$$

Then the vacuum expectation (4.6) can be obtained by applying the Murnaghan-Nakayama rule (2.5).

I'd like to mention that the operator formula (4.6) provides in some way a tool for calculating the relative GW-invariants over two points of $\mathbb{P}^1$, especially in 1-point case it is exactly the formula for working out the completion coefficients mentioned at the end of Section 1.3.

Remark. Since $\mathcal{P}_k |\lambda\rangle = \mathbf{p}_k(\lambda) |\lambda\rangle$, the related convention for $\mathcal{P}_k$ as $k \leq 0$ is given by

$$\mathcal{P}_0 = C, \quad \frac{1}{(-1)!}\mathcal{P}_{-1} = 1, \quad \mathcal{P}_k \equiv 0 \text{ otherwise.} \tag{4.7}$$

4.2.1. *τ-function in GW-theory.* Define the τ-function for GW-theory of $\mathbb{P}^1$ relative to 0 and ∞ to be

$$\tau(t, s; x) = \sum_{|\mu|=|\nu|} t_\mu s_\nu \left\langle \mu, \exp\left(\sum_{i \geq 0} x_i \tau_i(\omega) \right), \nu \right\rangle^{\bullet\mathbb{P}^1}. \tag{4.8}$$

By taking log, the connected theory will be recovered

$$\log \tau(t,s;x) = \sum_{|\mu|=|\nu|} t_\mu s_\nu \left\langle \mu, \exp\left(\sum_{i\geq 0} x_i \tau_i(\omega)\right), \nu \right\rangle^{\circ \mathbb{P}^1}.$$

Proposition 4.1.

$$\tau(t,s;x) = \left\langle \Gamma_+(t) \exp\left(\sum_{k\geq 0} \frac{x_k}{(k+1)!} \mathcal{P}_{k+1}\right) \Gamma_-(s) \right\rangle.$$

Proof. First, we extract the coefficients from the operator formula (4.6)

$$\left\langle \mu, \prod_{i=1}^{n} \tau_{k_i-1}(\omega), \nu \right\rangle^{\bullet \mathbb{P}^1} = \frac{1}{\mathfrak{z}(\mu)\mathfrak{z}(\nu)} \left\langle \prod_{i=1}^{l(\mu)} \alpha_{\mu_i} \prod_{i=1}^{n} [z_i^{k_i}] \mathcal{E}_0(z_i) \prod_{i=1}^{l(\nu)} \alpha_{-\nu_i} \right\rangle \tag{4.9}$$

$$= \frac{1}{\mathfrak{z}(\mu)\mathfrak{z}(\nu)} \left\langle \prod_{i=1}^{l(\mu)} \alpha_{\mu_i} \prod_{i=1}^{n} \frac{\mathcal{P}_{k_i}}{k_i!} \prod_{i=1}^{l(\nu)} \alpha_{-\nu_i} \right\rangle.$$

There's a general formula

$$\exp(\sum_{k>0} n_k) = \sum_{\text{partition } \lambda} \frac{1}{|\mathrm{Aut}\lambda|} \prod_{i=1}^{l(\lambda)} n_{\lambda_i}. \tag{4.10}$$

Now we can proceed to the calculation

$$\tau(t,s;x) = \sum_{|\mu|=|\nu|} t_\mu s_\nu \left\langle \mu, \exp\left(\sum_{i\geq 0} x_i \tau_i(\omega)\right), \nu \right\rangle^{\bullet \mathbb{P}^1}$$

$$\overset{(4.10)}{=} \sum_{|\mu|=|\nu|} t_\mu s_\nu \left\langle \mu, \sum_{\lambda} \frac{1}{|\mathrm{Aut}\lambda|} \prod_{i=1}^{l(\lambda)} x_{\lambda_i - 1} \tau_{\lambda_i - 1}(\omega), \nu \right\rangle^{\bullet \mathbb{P}^1}$$

$$\overset{(4.9)}{=} \sum_{|\mu|=|\nu|} \frac{t_\mu s_\nu}{\mathfrak{z}(\mu)\mathfrak{z}(\nu)} \left\langle \prod_{i=1}^{l(\mu)} \alpha_{\mu_i} \sum_{\lambda} \frac{1}{|\mathrm{Aut}\lambda|} \prod_{i=1}^{l(\lambda)} \frac{x_{\lambda_i - 1}}{\lambda_i!} \mathcal{P}_{\lambda_i} \prod_{i=1}^{l(\nu)} \alpha_{-\nu_i} \right\rangle$$

$$\overset{(4.10)}{=} \sum_{|\mu|=|\nu|} \frac{t_\mu s_\nu}{\mathfrak{z}(\mu)\mathfrak{z}(\nu)} \left\langle \prod_{i=1}^{l(\mu)} \alpha_{\mu_i} \exp\left(\sum_{k=0}^{\infty} \frac{x_k}{(k+1)!} \mathcal{P}_{k+1}\right) \prod_{i=1}^{l(\nu)} \alpha_{-\nu_i} \right\rangle$$

$$= \left\langle \sum_{\mu} \frac{t_\mu}{\mathfrak{z}(\mu)} \prod_{i=1}^{l(\mu)} \alpha_{\mu_i} \exp\left(\sum_{k=0}^{\infty} \frac{x_k}{(k+1)!} \mathcal{P}_{k+1}\right) \sum_{\nu} \frac{s_\nu}{\mathfrak{z}(\nu)} \prod_{i=1}^{l(\nu)} \alpha_{-\nu_i} \right\rangle$$

(note that the constraint $|\mu| = |\nu|$ is actually redundant.)

$$= \left\langle \Gamma_+(t) \exp\left(\sum_{k\geq 0} \frac{x_k}{(k+1)!} \mathcal{P}_{k+1}\right) \Gamma_-(s) \right\rangle.$$

□

4.2.2. *String equations.* In either connected or disconnected case, we have the following *string equation*

$$\left\langle \tau_0(1) \prod_{i=1}^{n} \tau_{k_i}(\omega) \right\rangle = \sum_{j=1}^{n} \left\langle \prod_{i=1}^{n} \tau_{k_i - \delta_{i,j}}(\omega) \right\rangle,$$

from which we can get

$$\left\langle e^{y\tau_0(1)} \prod_{i=1}^{n} \tau_{k_i}(\omega) \right\rangle = \left\langle \prod_{i=1}^{n} \left(\sum_{m \geq 0} \frac{y^m}{m!} \tau_{k_i - m}(\omega) \right) \right\rangle. \tag{4.11}$$

Indeed, the insertion of the right hand side is

$$\sum_{m_1,\ldots,m_n \geq 0} \frac{y^{m_1+\cdots+m_n}}{m_1! \cdots m_n!} \tau_{k_1 - m_1}(\omega) \cdots \tau_{k_n - m_n}(\omega),$$

and the only way to obtain the term $\tau_{k_1-m_1}(\omega) \cdots \tau_{k_n-m_n}(\omega)$ from the left hand side is to apply $\frac{y^m}{m!}\tau_0(1)^m$ with $m = m_1 + \ldots + m_n$. By the string equation, there will be $\binom{m}{m_1,\ldots,m_n}$ many $\prod_{i=1}^{n} \tau_{k_i - m_i}(\omega)$'s leaved after the action of $\tau_0(1)^m$. Hence the coefficient of this term is $\frac{y^m}{m!}\binom{m}{m_1,\ldots,m_n} = \frac{y^m}{m_1!\cdots m_n!}$, which fits exactly the right hand side.

The string equation can be further rewritten as a generating series

$$\sum_{k_i=-\infty}^{\infty} \left\langle e^{y\tau_0(1)} \prod_{i=1}^{n} \tau_{k_i}(\omega) \right\rangle \prod_{i=1}^{n} z_i^{k_i+1} = e^{\sum_{i=1}^{n} yz_i} \sum_{k_i=\infty}^{\infty} \left\langle \prod_{i=1}^{n} \tau_{k_i}(\omega) \right\rangle \prod_{i=1}^{n} z_i^{k_i+1}, \tag{4.12}$$

which follows from

$$\begin{aligned}
\sum_{k_i=-\infty}^{\infty} \left\langle e^{y\tau_0(1)} \prod_{i=1}^{n} \tau_{k_i}(\omega) \right\rangle \prod_{i=1}^{n} z_i^{k_i+1} &= \sum_{k_i=-\infty}^{\infty} \left\langle \prod_{i=1}^{n} \left(\sum_{m \geq 0} \frac{y^m}{m!} \tau_{k_i-m}(\omega) \right) \right\rangle \prod_{i=1}^{n} z_i^{k_i+1} \\
&= \left\langle \prod_{i=1}^{n} \sum_{m \geq 0} \frac{y^m z_i^m}{m!} \underbrace{\sum_{k_i=-\infty}^{\infty} \tau_{k_i-m}(\omega) z_i^{k_i-m+1}}_{\text{(independent of m)}} \right\rangle \\
&= \exp\left(\sum_{i=1}^{n} yz_i \right) \sum_{k_i=\infty}^{\infty} \left\langle \prod_{i=1}^{n} \tau_{k_i}(\omega) \right\rangle \prod_{i=1}^{n} z_i^{k_i+1}.
\end{aligned}$$

4.2.3. *τ-function with string variable.* Now we plus a *string variable y* into the τ-function:

$$\tau_y(t, s; x) = \sum_{|\mu|=|\nu|} t_\mu s_\nu \left\langle \mu, \exp\left(y\tau_0(1) + \sum_{i \geq 0} x_i \tau_i(\omega) \right), \nu \right\rangle^{\bullet \mathbb{P}^1}. \tag{4.13}$$

Similarly, we have

$$\log \tau_y(t,s;x) = \sum_{|\mu|=|\nu|} t_\mu s_\nu \left\langle \mu, \exp\left(y\tau_0(1) + \sum_{i\geq 0} x_i \tau_i(\omega) \right), \nu \right\rangle^{\circ \mathbb{P}^1}.$$

Proposition 4.2. *For $n \in \mathbb{Z}$, we have*

$$\tau_n(t,s;x) = \left\langle T^{-n}\Gamma_+(t) \exp\left(\sum_{k\geq 0} \frac{x_k}{(k+1)!} \mathcal{P}_{k+1} \right) \Gamma_-(s) T^n \right\rangle.$$

Proof. Except for some additional work caused by the appearance of the string variable, the calculation mainly follows the previous case.

By definition, $T^{-n}\mathcal{E}_0(z)T^n = e^{zn}\mathcal{E}_0(z)$. Together with the convention (4.7), the operators T and $\mathcal{P}_k$ satisfy the following commutation relation

$$T^{-n}\frac{\mathcal{P}_k}{k!}T^n = \sum_{m=0}^{k+1} \frac{n^m}{m!} \frac{\mathcal{P}_{k-m}}{(k-m)!}. \tag{4.14}$$

Then

$$\begin{aligned}
\tau_n(t,s;x) &= \sum_{|\mu|=|\nu|} t_\mu s_\nu \left\langle \mu, e^{n\tau_0(1)} \exp\left(\sum_{i\geq 0} x_i \tau_i(\omega) \right), \nu \right\rangle^{\bullet \mathbb{P}^1} \\
&\overset{(4.10)}{=} \sum_{|\mu|=|\nu|} t_\mu s_\nu \sum_\lambda \frac{\prod_{i=1}^{l(\lambda)} x_{\lambda_i - 1}}{|\mathrm{Aut}\lambda|} \left\langle \mu, e^{n\tau_0(1)} \prod_{i=1}^{l(\lambda)} \tau_{\lambda_i - 1}(\omega), \nu \right\rangle^{\bullet \mathbb{P}^1} \\
&\overset{(4.11)}{=} \sum_{|\mu|=|\nu|} t_\mu s_\nu \sum_\lambda \frac{\prod_{i=1}^{l(\lambda)} x_{\lambda_i - 1}}{|\mathrm{Aut}\lambda|} \left\langle \mu, \prod_{i=1}^{l(\lambda)} \left(\sum_{m=0}^{\lambda_i+1} \frac{n^m}{m!} \tau_{\lambda_i-1-m}(\omega) \right), \nu \right\rangle^{\bullet \mathbb{P}^1} \\
&\overset{(4.9)}{=} \sum_{|\mu|=|\nu|} \frac{t_\mu s_\nu}{\mathfrak{z}(\mu)\mathfrak{z}(\nu)} \sum_\lambda \frac{\prod_{i=1}^{l(\lambda)} x_{\lambda_i - 1}}{|\mathrm{Aut}\lambda|} \left\langle \prod_{i=1}^{l(\mu)} \alpha_{\mu_i} \prod_{i=1}^{l(\lambda)} \left(\sum_{m=0}^{\lambda_i+1} \frac{n^m}{m!} \frac{\mathcal{P}_{\lambda_i - m}}{(\lambda_i - m)!} \right) \prod_{i=1}^{l(\nu)} \alpha_{-\nu_i} \right\rangle \\
&\overset{(4.14)}{=} \sum_{|\mu|=|\nu|} \frac{t_\mu s_\nu}{\mathfrak{z}(\mu)\mathfrak{z}(\nu)} \sum_\lambda \frac{\prod_{i=1}^{l(\lambda)} x_{\lambda_i - 1}}{|\mathrm{Aut}\lambda|} \left\langle \prod_{i=1}^{l(\mu)} \alpha_{\mu_i} \prod_{i=1}^{l(\lambda)} \left(T^{-n} \frac{\mathcal{P}_{\lambda_i}}{\lambda_i!} T^n \right) \prod_{i=1}^{l(\nu)} \alpha_{-\nu_i} \right\rangle \\
&\overset{(4.10)}{=} \sum_{|\mu|=|\nu|} \frac{t_\mu s_\nu}{\mathfrak{z}(\mu)\mathfrak{z}(\nu)} \left\langle T^{-n} \prod_{i=1}^{l(\mu)} \alpha_{\mu_i} \exp\left(\sum_{k\geq 0} x_k \frac{\mathcal{P}_{k+1}}{(k+1)!} \right) \prod_{i=1}^{l(\nu)} \alpha_{-\nu_i} T^n \right\rangle \\
&= \left\langle T^{-n}\Gamma_+(t) \exp\left(\sum_{k\geq 0} \frac{x_k}{(k+1)!} \mathcal{P}_{k+1} \right) \Gamma_-(s) T^n \right\rangle.
\end{aligned}$$

□

4.2.4. *The Toda hierarchy.* Since $[\mathcal{E}_0(z), \psi_k] = 0 = [\mathcal{E}_0(z), \psi_k^*]$, the operator $M := \sum_{k\geq 0} \frac{x_k}{(k+1)!}\mathcal{P}_{k+1}$ satisfies the condition (4.1). From the above proposition $\tau_n(t, s; x)$ coincides with the τ-function introduced first in this section, so we can conclude that

Theorem 4.3. *The τ-function $\{\tau_n(t, s; x)\}$ satisfies the Hirota equations for the Toda hierarchy. In particular it satisfies the lowest degree equation:*

$$\frac{\partial^2}{\partial t_1 \partial s_1} \log \tau_n = \frac{\tau_{n+1}\tau_{n-1}}{\tau_n^2}. \tag{4.15}$$

Where $\tau_n = \tau_n(t, s; x)$.

The special case (4.15) can be rewritten in the following form

Proposition 4.4.

$$\mathsf{F}^\circ_{\mu+1,\nu+1}(z_1, \ldots, z_n) =$$
$$\frac{1}{(m_1(\mu)+1)(m_1(\nu)+1)} \sum_{\{(S_l,\mu^l,\nu^l)_l\}\in \mathrm{Part}[n,\mu,\nu]} \prod_l \varsigma\left(\sum_{i\in S_l} z_i\right)^2 \mathsf{F}^\circ_{\mu^l,\nu^l}(z_{S_l}),$$

where the triples $\{(S_i, \mu^i, \nu^i)\}$ consists of

(i) $\{S_l\}$ *is a partition of the set* $\{1, \ldots, n\}$, $S_l \neq \emptyset$.
(ii) $\{\mu^l\}$ *is a partition of the partition* μ, $\mu = \bigcup \mu^l$; *similar for* $\{\nu^l\}$ *with* $|\mu^l| = |\nu^l|$.

Proof. Observe that the formula (4.10) has an alternating form: if we index the right hand side by *components* $\kappa = (k_1, \ldots, k_{l(\kappa)})$, i.e. ordered partitions, we'll get

$$\exp(\sum_{k>0} n_k) = \sum_{\kappa} \frac{1}{l(\kappa)!} \prod_{i=1}^{l(\kappa)} n_{k_i}. \tag{4.16}$$

Here the ordering of the product $\prod_{i=1}^{l(\kappa)} n_{k_i}$ is concerned. This formula could be generalized to some other index set, for example, the *double partitions* $\{k = (\mu, \nu)\}$.

Let's compare the coefficient of $t_\mu s_\nu$ from both sides of (4.15) with $n = 0$: For the left hand side

$$
\begin{aligned}
&[t_\mu s_\nu]\frac{\partial^2}{\partial t_1 \partial s_1}\log\tau_0 \\
&= (m_1(\mu)+1)(m_1(\nu)+1)\left\langle \mu+1, \exp\left(\sum_{i\geq 0} x_i\tau_i(\omega)\right), \nu+1 \right\rangle^{\circ\mathbb{P}^1} \\
&\overset{(4.16)}{=} (m_1(\mu)+1)(m_1(\nu)+1)\sum_{\kappa}\frac{x_\kappa}{l(\kappa)!}\left\langle \mu+1, \prod_{i=1}^{l(\kappa)}\tau_{k_i}(\omega), \nu+1 \right\rangle^{\circ\mathbb{P}^1}.
\end{aligned}
$$

For the right hand side

$$
\begin{aligned}
[t_\mu s_\nu]\frac{\tau_1\tau_{-1}}{\tau_0^2} &= [t_\mu s_\nu]\exp\left(\sum_{\mu',\nu'} t_{\mu'}s_{\nu'}\left\langle \mu', \varsigma(\tau_0(1))^2\exp\left(\sum_{i\geq 0}x_i\tau_i(\omega)\right), \nu'\right\rangle^{\circ\mathbb{P}^1}\right) \\
&\overset{(4.16)}{=} \sum_{\substack{\{(\mu^l,\nu^l):l=1,\ldots,N\}\\ \in\mathrm{Part}[\mu,\nu]}} \frac{1}{N!}\prod_{l=1}^{N}\left\langle \mu^l, \varsigma(\tau_0(1))^2\exp\left(\sum_{i\geq 0}x_i\tau_i(\omega)\right), \nu^l\right\rangle^{\circ\mathbb{P}^1} \\
&\overset{(4.16)}{=} \sum_{\substack{\{(\mu^l,\nu^l):l=1,\ldots,N\}\\ \in\mathrm{Part}[\mu,\nu]}} \frac{1}{N!}\prod_{l=1}^{N}\sum_{\kappa^l}\frac{x_{\kappa^l}}{l(\kappa^l)!}\left\langle \mu^l, \varsigma(\tau_0(1))^2\prod_{i=1}^{l(\kappa^l)}\tau_{k_i}(\omega), \nu^l\right\rangle^{\circ\mathbb{P}^1}.
\end{aligned}
$$

Now we can compare the both sides in coefficients x_κ. For simplicity, we let $l(\kappa) = n$, $l(\kappa^l) = n_l$ in what follows.

$$
\begin{aligned}
(4.17)\quad & (m_1(\mu)+1)(m_1(\nu)+1)\frac{1}{n!}\left\langle \mu+1, \prod_{i=1}^{n}\tau_{k_i}(\omega), \nu+1 \right\rangle^{\circ\mathbb{P}^1} \\
&= \sum_{\substack{\{(\mu^l,\nu^l,\kappa^l):l=1,\ldots,N\}\\ \in\mathrm{Part}[\mu,\nu,\kappa]}} \frac{1}{N!n_1!\cdots n_N!}\prod_{l=1}^{N}\left\langle \mu^l, \varsigma(\tau_0(1))^2\prod_{i=1}^{n_l}\tau_{k_i}(\omega), \nu^l\right\rangle^{\circ\mathbb{P}^1} \\
&\quad\times\left(\frac{n!}{N!n_1!\cdots n_N!}\right)^{-1},
\end{aligned}
$$

where the factor $\left(\frac{n!}{N!n_1!\cdots n_N!}\right)^{-1}$ is introduced to eliminate the overcountings.

Recall that $\varsigma(z) = e^{z/2} - e^{-z/2}$. By (4.12) we have the following equation

$$\sum_{k_i} \left\langle \mu, \varsigma(\tau_0(1))^2 \prod_{i=1}^{n} \tau_{k_i}(\omega), \nu \right\rangle \prod_{i=1}^{n} z_i^{k_i+1} = \varsigma\left(\sum_{i=1}^{n} z_i\right)^2 \sum_{k_i} \left\langle \mu, \prod_{i=1}^{n} \tau_{k_i}(\omega), \nu \right\rangle \prod_{i=1}^{n} z_i^{k_i+1}. \tag{4.18}$$

By using this, the above formula can be carried out as

$$\begin{aligned}
&(m_1(\mu)+1)(m_1(\nu)+1)\mathsf{F}^{\circ}_{\mu+1,\nu+1}(z_1,\ldots,z_n)\\
&= (m_1(\mu)+1)(m_1(\nu)+1)\sum_{k_i}\left\langle \mu+1, \prod_{i=1}^{n}\tau_{k_i}(\omega), \nu+1\right\rangle^{\circ\mathbb{P}^1}\prod_{i=1}^{n} z_i^{k_i+1}\\
&\overset{(4.17)}{=} \sum_{\substack{\{(S_l,\mu^l,\nu^l):l=1,\ldots,N\}\\ \in\mathrm{Part}[n,\mu,\nu]}} \prod_{l=1}^{N}\left(\sum_{k_i\in S_l}\left\langle \mu^l, \varsigma(\tau_0(1))^2\prod_{i\in S_l}\tau_{k_i}(\omega), \nu^l\right\rangle^{\circ\mathbb{P}^1}\prod_{i\in S_l} z_i^{k_i+1}\right)\\
&\overset{(4.18)}{=} \sum_{\substack{\{(S_l,\mu^l,\nu^l):l=1,\ldots,N\}\\ \in\mathrm{Part}[n,\mu,\nu]}} \prod_{l=1}^{N}\varsigma\left(\sum_{i\in S_l} z_i\right)^2\left(\sum_{k_i\in S_l}\left\langle \mu^l, \prod_{i\in S_l}\tau_{k_i}(\omega), \nu^l\right\rangle^{\circ\mathbb{P}^1}\prod_{i\in S_l} z_i^{k_i+1}\right)\\
&= \sum_{\substack{\{(S_l,\mu^l,\nu^l):l=1,\ldots,N\}\\ \in\mathrm{Part}[n,\mu,\nu]}} \prod_{l=1}^{N}\varsigma\left(\sum_{i\in S_l} z_i\right)^2 \mathsf{F}^{\circ}_{\mu^l,\nu^l}(z_{S_l}),
\end{aligned}$$

as required. □

Remark. From the proof it can be found that the (4.15) is also equivalent to

$$(m_1(\mu)+1)(m_1(\nu)+1)\left\langle \mu+1, \prod_{i=1}^{l(\kappa)}\tau_{k_i}(\omega), \nu+1\right\rangle^{\circ\mathbb{P}^1} = \left\langle \mu, \varsigma(\tau_0(1))^2\prod_{i=1}^{l(\kappa)}\tau_{k_i}(\omega), \nu\right\rangle^{\bullet\mathbb{P}^1}.$$

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