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瑞曲流孤立子

A Survey on Gradient Ricci Solitons

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中文摘要

瑞曲流 (Ricci flow) 為理察 • 哈密頓 (Richard Hamilton) 為解決三維龐加萊猜想 (Poincaré conjecture) 所發展的重要工具。瑞曲流中的孤立子 (Solitons) 是在瑞曲流中的自我相似解 (self-similar solution)，是瑞曲流奇點的重要模型，裴瑞爾曼 (Grigori Perelman) 在三維成功發展處理孤立子的技巧，進而解決龐加萊猜想。這些孤立子的分類中，有一類稱為梯度孤立子 (Gradient soliton)，可由梯度函數描述。

在 2015 年 Ovidiu Munteanu 與王嘉平共同發表的一篇論文中，展示了一種估計四維瑞曲流梯度孤立子中黎曼曲率、里奇曲率與純量曲率的方法，本論文將介紹前人在多維度梯度孤立子的一些結果，並介紹 Ovidiu Munteanu 與王嘉平在四維上的估計方法。

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英文摘要

To solve the Poincaré conjecture on 3-dimensional cases, Richard Hamilton evolved an algorithm called Ricci flow. In Ricci flow, a class of self-similar solutions called gradient solitons. Studing of such kink solution is playing an important role in solving Poincaré conjecture.

In 2015, Ovidiu Munteanu and Jiaping Wang shown an algorithm to estimate the Riemann, Ricci curvature and scalar curvature on 4-dimansional gradient solitons in Ricci flow. In this survey, I would introduce some early results in gradient solitons and explore the details in Ovidiu Munteanu and Jiaping Wang's paper in 4-dimensional shrinking solitons.

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1 Introduction

Let (M, g) be a complete manifold and consider the metric g evolving by time variable t . The Ricci flow is a geometry flow defined by the equation

$$\partial_t g_{ij}(t) = -2R_{ij},$$

where R_{ij} is Ricci curvature tensor and g_{ij} is metric tensor depending on t. A solution of Ricci flow called Ricci soliton if there is a vector filed $X(t)$ satisfied

$$R_{ij} + L_X g_{ij} = \mu g_{ij},$$

where L_X is Lie derivative and μ is a constant. Moreover, if there exist a function f such that $\nabla f = X$ then we can define gradient Ricci solitons as follow:

Definition 1.1. *(M, g, f) is called gradient Ricci soliton if $R_{ij} + \nabla_i \nabla_j f = \mu g_{ij}$. such soliton is called shrinking, steady and expanding if $\mu > 0, \mu = 0$ and $\mu < 0$ respectly.*

Without loss of generally, we can rescale metric g and customarily let $\mu = 1/2, \mu = 0$ and $\mu = 1/2$ in each case. In this survey we would concern mainly on shrinking cases. Those solitons take an important part in the singularity analysis of Ricci flow.J. Enders, et al. shows that blow up around a type-I singularity point in 3-dimensional Ricci flow always converge to shrinking gradient Ricci solitons [5] . Thus a brief analysis in 4-dimensional cases would be a central issue in Ricci flow.

In three-dimensional shrinking gradient Ricci solitons. The curvature operator is being nonnegetive and bounded by scalar curvature, but it is no longer be true in higher dimensional cases. However, in this paper, O. Munteanu and J. Wang show that in four-dimensional cases, if we assume scalar curvature R is bounded, then the curvature operator is bounded by scalar and bounded blow by zero when the distence r goes to infinite [1].

Theorem 1.1. *Let (M, g, f) be four dimensional shrinking soliton with bounded scalar*

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curvature $0 < R < A$. *then*

$$|Rm| \leq c_1 R$$

and

$$Rm \geq -(\frac{c_2}{\log(r(x)+1)})^{\frac{1}{4}},$$

where c_1 *and* c_2 *are constants depanding on* A *and on metric* g *within a geodesic ball* $B_p(\gamma_0)$, p *is minimun point of* f *and* γ_0 *is only depending on* A, $r(x) := d(x,p)$ *is distence function.*

There left some problems in this state of theorem, like how about the cases with $R \leq 0$ and the existence of the minimum point of f. We would discuss briefly in next section.

2 Preliminary Results in Gradient Ricci Solotion

In this thesis, we rescale the metric g and define

Definition 2.1. (M, g, f) *is called shrinking gradient Ricci soliton if*

$$R_{ij} + \nabla_i \nabla_j f = \frac{1}{2} g_{ij}.$$

One would easily get

Proposition 2.1. *Let* (M, g, f) *be a shrinking gradient Ricci soliton, then*

$$R + |\nabla f|^2 - f = constant$$

on M.

For convenience, we can add the constant to f and let

$$R + |\nabla f|^2 = f.$$

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With those assumptions, if we define the weighted Laplacian $\Delta_f := \Delta - \langle \nabla f, \nabla \rangle$, we can compute the fallowing proposition.

Proposition 2.2. *In four-dimensional shrinking gradient Ricci soliton, we have*

$$\begin{aligned}
\Delta_f f &= \frac{n}{2} - f, \\
\Delta_f R &= R - 2|Ric|^2, \\
\Delta_f R_{ij} &= R_{ij} - 2R_{iklj}R_{kl}, \\
\Delta_f Rm &= Rm + Rm * Rm, \\
\nabla_k R_{jk} &= R_{jk} f_k = \frac{1}{2}\nabla_j R, \\
\nabla_l R_{ijkl} &= R_{ijkl} f_l = \nabla_i R_{jk} - \nabla_j R_{ik}, \\
\Delta_f \nabla S - \nabla \Delta_f S &= \frac{1}{2}\nabla S + Rm * \nabla S \quad for\ any\ tensor\ S.
\end{aligned}$$

One could see more detail in [6]. Concerning the scalar curvature R, B.L. Chen have proved that

Theorem 2.1 (B.L. Chen[3])**.** *If (M, g, f) is shrinking gradient Ricci soliton, Then scalar curvature $R > 0$ unless M is flat.*

Concernig on the potential function f,Cao and Zhou have proved that

Theorem 2.2 (H. D. Cao and D. Zhou[2])**.** *If (M, g, f) is shrinking gradient Ricci soliton, Then*

$$(Max\{0, \frac{1}{2}r(x) - c_1\})^2 \leq f(x) \leq (\frac{1}{2}r(x) - c_2)^2$$

Where $r(x) := d(p, x)$ is distance from x to p, the minimum point of f. Constants c_1 and c_2 can be choosen to depand only on dimension n.

Based on theorem 2.1 and theorem 2.2. B. Chow, P. Lu and B. Yang can show that

Theorem 2.3 (B. Chow, R. Lu and B. Yang[4])**.** *If (M, g, f) is shrinking gradient Ricci*

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soliton, Then there exist canstant $C > 0$ such that

$$C \leq fR.$$

Since the quadraticaly growth of f, one can define

Definition 2.2.

$$D(t) = \{x \in M | f(x) \leq t\}$$

$$\Sigma(t) = \{x \in M | f(x) = t\}.$$

It is clear that both $D(t)$ and $\Sigma(t)$ is compact.note that when $R \leq A$, if there exist some constants $\gamma_0 \geq \frac{A}{3}$, then

$$1 \leq \frac{1}{2}\sqrt{f} \leq \sqrt{f - A} \leq \sqrt{f - R} \leq |\nabla f| \; on \; M \setminus D(\gamma_0),$$

hence

$$1 \leq \frac{1}{2}\sqrt{f} \leq |\nabla f| \quad on \; M \setminus D(\gamma_0).$$

3 Maximum principle on shrinking gradient Ricci solitons

Before we go to the main theory, here we state some useful theorems of maximun principle in four-dimensional gradient Ricci solitons with bounded scalar curvature $R < A$.

Theorem 3.1. *Let (M, g, f) be a four-dimensional shrinking gradient Ricci soliton with bounded scalar curvature $R \leq A$, and $u > 0$ be an smooth function on M. For $m > 1$, if there exist an constant γ_0 such that*

$$\Delta_f u \geq c_m u^m + l.o.t \; of \; u + \langle \nabla u, \nabla F \rangle$$

on $M \setminus D(\gamma_0)$, where c_i are smooth functions difined on M with leader coefficient $c_m > 0$ and $\frac{|c_i|}{c_m}$ is bounded by constants k_i on $M \setminus D(\gamma_0)$ for all $i < m$. F is an smooth function

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satisfied $\frac{|\nabla F|}{\sqrt{f}}$ *converging to zero at infinity, then*

$$\sup_{M\setminus D(2\gamma_0)} u < C,$$

note that the upper bound C *depends only on* k_i *for* $i < m$. *In particular,*

$$\sup_M u < C.$$

Proof. First we assume that γ_0 is big enough such that $|\nabla F| \leq \sqrt{f}$ let $\phi : \mathbb{R} \to \mathbb{R}$ be a smooth non-negtive function so that $\phi = 1$ on $[\gamma, 2\gamma]$ and $\phi = 0$ outside $[\gamma/2, 3\gamma]$, we could suppose that c is big enough such that

$$t^2(|\phi'|^2(t) + |\phi''|(t)) \leq c.$$

Note that the choosen of c is independent of γ. It fallows that

$$|\nabla\phi(f)|^2 \leq \frac{c}{\gamma}, |\Delta_f\phi(f)| \leq c$$

on $M \setminus D(\gamma_0)$. Let $G := u\phi^2$ and choose $\gamma \geq 2\, max\{1, \gamma_0\}$ we compute:

$$\begin{aligned}
\phi^{2m-2}\Delta_f G &= \phi^{2m}\Delta_f u + \phi^{2m-4}G\Delta_f\phi^2 + 2\phi^{2m-4}\langle\nabla G, \nabla\phi^2\rangle \\
&\quad -8\phi^{2m-4}G|\nabla\phi|^2 \\
&\geq \phi^{2m}\Delta_f u + \phi^{2m-4}GG\Delta_f\phi^2 + 2\phi^{2m-4}G\langle\nabla G, \nabla\phi^2\rangle - c\frac{G}{\gamma} \\
&\geq \phi^{2m}\Delta_f u - cG + 2\langle\nabla G, \nabla\phi^2\rangle, \\
\phi^{2m}\Delta_f u &\geq \phi^{2m}(c_m u^m + l.o.t\ of\ u + \langle\nabla u, \nabla F\rangle) \\
&\geq c_m G^m + \phi^{2m}(l.o.t\ of\ u) + \phi^{2m-2}\langle\nabla G, \nabla F\rangle \\
&\quad -\phi^{2m-4}G\langle\nabla\phi^2, \nabla F\rangle, \\
-\phi^{2m-4}G\langle\nabla\phi^2, \nabla F\rangle &\geq -G\phi^{2m-3}\frac{c}{\sqrt{\gamma}}\sqrt{f} \\
&\geq -cG.
\end{aligned}$$

Note that if $0 \leq a < m$, then $\phi^{2m}c_a u^a = \phi^{2m-2a}c_a G^a \geq -k_i G^a$ on $M \setminus D(2\gamma_0)$. Hence

$$\phi^{2m}(l.o.t\ of\ u) \geq l.o.t\ of\ G.$$

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Then we get

$$\phi^{2m-2}\Delta_f G \geq c_0 G^m + l.o.t\ of\ G + 2\phi^{2m-4}\langle \nabla G, \nabla \phi^2 \rangle + \phi^{2m-2}\phi^2 \langle \nabla G, \nabla F \rangle.$$

Note that the coefficients on right hand side are noly depend on k_i, hence by maximun principle we proove the theorem. □

Remark that theorem 3.1 is fail when $m = 1$. In such case, we have the fallowing theorem.

Theorem 3.2. *Let (M, g, f) be a four-dimensional shrinking gradient Ricci soliton with bounded scalar curvature $R \leq A$, and $u > 0$ be an function on M if there exist an constant γ_0 such that*

$$\Delta_f u \geq c_1 u + c_0 + \langle \nabla u, \nabla F \rangle$$

and

$$u \leq cf^2$$

on $M \setminus D(\gamma_0)$, where c_1 and c_0 are constants. F is an smooth function satisfied $\frac{|\nabla F|}{\sqrt{f}}$ converging to zero at infinity, then

$$\sup_M u < C$$

.

Proof. let $\psi(t) := 1 - \frac{t}{\gamma}$ then

$$|\nabla \psi(f)|^2 \leq \frac{|\nabla f|^2}{\gamma^2} \leq \frac{1}{\gamma}, \Delta_f \psi(f) = \frac{f-2}{\gamma}$$

on $M \setminus D(\gamma_0)$. Now let $G := \psi^2 u$ and choose $\gamma \geq 2\gamma_0$ we compute:

Case (a). G achive maximun in $D(\gamma_0)$

Then we have

$$\sup_{D(\gamma/2)} u \leq 4 \sup_{D(\gamma/2)} G \leq 4 \sup_{D(\gamma_0)} G \leq C$$

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Case (b). G achive maximun in $M \setminus D(\gamma_0)$

Direct computation gives that

$$\begin{aligned}
\Delta_f G &= \psi^2\Delta_f u + \psi^{-2}G\Delta_f\psi^2 + 2\psi^{-2}\langle\nabla G, \nabla\psi^2\rangle - 8\psi^{-2}G|\nabla\psi|^2 \\
&= \psi^2(c_1 u + c_0 + \langle\nabla u, \nabla F\rangle) + 2\psi^{-2}\langle\nabla G, \nabla\psi^2\rangle \\
&\quad +2\psi^{-1}G\Delta_f\psi - 6\psi^{-2}G|\nabla\psi|^2 \\
&= \psi^2(c_1 u + c_0) + \langle\nabla G, \nabla F\rangle - \psi^{-2}G\langle\nabla\psi^2, \nabla F\rangle \\
&\quad +2\psi^{-2}\langle\nabla G, \nabla\psi^2\rangle + 2\psi^{-1}G\Delta_f\psi - 6\psi^{-2}G|\nabla\psi|^2 \\
&\geq (c_1 G - c_0) + \langle\nabla G, \nabla(4\ln\psi + F)\rangle \\
&\quad -2\psi^{-1}G\langle\nabla\psi, \nabla F\rangle + 2\psi^{-1}G\Delta_f\psi - 6\psi^{-2}G|\nabla\psi|^2
\end{aligned}$$

In the line four we use the same argumente as theorem 3.1 did. Note that coefficients of lower order terms is negative in line four.Let $q \in M \setminus D(\gamma_0)$ be the maximun point of G, then at q

$$0 \geq (c_1 G - c_0) - 2\psi^{-1}G\langle\nabla\psi, \nabla F\rangle + 2\psi^{-1}G\Delta_f\psi - 6\psi^{-2}G|\nabla\psi|^2.$$

If $c_1 G(q) - c_0 \leq 0$, then $G(q) \leq c_0$. we can conclude that $G(q)$ is bound by constant independ of γ, and $\sup\limits_{D(\gamma/2)} u \leq 4 \sup\limits_{D(\gamma/2)} G \leq G(q) \leq C$. So we can assume $c_1 G(q) - c_0 \geq 0$, then

$$0 \geq -2\psi^{-1}G\langle\nabla\psi, \nabla F\rangle + 2\psi^{-1}G\Delta_f\psi - 6\psi^{-2}G|\nabla\psi|^2.$$

Suppose that γ_0 is big enough such that $|\nabla F| \leq \frac{1}{4}\sqrt{f}$ and $\frac{1}{2}f - 2 \geq \frac{1}{4}f$, we can estimate

$$\begin{aligned}
0 &\geq -2\psi^{-1}\langle\nabla\psi, \nabla F\rangle + 2\psi^{-1}\Delta_f\psi - 6\psi^{-2}|\nabla\psi|^2 \\
&\geq 2\psi^{-1}(\Delta_f\psi - \langle\nabla\psi, \nabla F\rangle) - 6\psi^{-2}\frac{|\nabla f|^2}{\gamma^2} \\
&\geq 2\psi^{-1}\frac{1}{\gamma}(f - 2 - \frac{1}{2}f) - 6\psi^{-2}\frac{f}{\gamma^2} \\
&\geq \psi^{-1}\frac{1}{\gamma}\frac{f}{2} - 6\psi^{-2}\frac{f}{\gamma^2}
\end{aligned}$$

at q. At the last line, we get $\psi(q) \leq \frac{12}{\gamma}$, hence

$$\sup_{D(\gamma/2)} u \leq 4 \sup_{D(\gamma/2)} G \leq 4G(q) \leq \frac{576}{\gamma^2}\sup_{D(\gamma)} u \leq C$$

by $u \leq cf^2$. $\square$

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4 Constant Upper bound of Curvature

In this section, we would show that in four-dimensional shrinking gradient Ricci solitons, if scalar curvature R has bounded, then Riemann curvature and it's derivation is also bounded by constants. Note that is not our main theory, but those are essential steps for the proof.

Before we start this section, remark that the shrinking gradient Ricci soliton we mention here been assumed with following properties:

$$R_{ij} + \nabla_i \nabla_j f = \frac{1}{2} g_{ij}$$

$$R + |\nabla f|^2 = f.$$

Lemma 4.1. *Let (M, g, f) be a four-dimensional shrinking gradient Ricci soliton with bounded scalar curvature $R \leq A$. Then there exist constants $C(A)$ and $\gamma_0(A)$ such that*

$$|Rm| \leq c(\frac{|\nabla Ric|}{\sqrt{f}} + |Ric|)$$

on $M \setminus D(\gamma_0)$.

Proof. Choose an orthonormal basis $\{e_i\}$ with $\{e_1, e_2, e_3\}$ locate on $T\Sigma(t)$, which t is big enough such that Σ forms a submanifold, and $e_4 = \frac{\nabla f}{|\nabla f|}$. Thanks the idea to Chih-Wei Chen, we note that

$$R_{1212} = -R_{11} - R_{22} + \frac{R}{2} + R_{3434}, \quad R_{1223} = R_{13} + R_{1434},$$
$$R_{2323} = -R_{22} - R_{33} + \frac{R}{2} + R_{1414}, \quad R_{2331} = R_{21} + R_{2414},$$
$$R_{3131} = -R_{11} - R_{33} + \frac{R}{2} + R_{2424}, \quad R_{1231} = R_{23} + R_{2434}.$$

We can estimate that

$$|R_{ijk4}| = \frac{1}{|\nabla f|} |R_{ijkl} f_l| \leq 4 \frac{|\nabla Ric|}{|\nabla f|}$$

for $i, j, k \in \{1, 2, 3, 4\}$, hence

$$|Rm| \leq c(|R_{ijkl}| + |R_{ijk4}|) \leq c(|Ric| + \frac{|\nabla Ric|}{|\nabla f|}).$$

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Since $R \leq A$, we can choose γ_0 big enough such that $1 \leq \frac{1}{2}\sqrt{f} \leq |\nabla f|$ on $M \setminus D(\gamma_0)$, we get

$$|Rm| \leq c(|Ric| + \frac{|\nabla Ric|}{\sqrt{f}}) \ on\ M \setminus D(\gamma_0).$$

□

Lemma 4.2. *Let (M, g, f) be a four-dimensional shrinking gradient Ricci soliton with bounded scalar curvature $R \leq A$. Then there exist constants C such that*

$$\sup_M \frac{|Ric|^2}{R} \leq C.$$

Proof. Let $u := |Ric|^2 R^{-a}$ where $0 < a < 1$. We first show that

$$\Delta_f u \geq (2a - \frac{cR}{(1-a)f})u^2 R^{-a} - cu^{3/2} S^{a/2} \ \ on\ M \setminus D(\gamma_0)$$

ues lemma 4.1, one get

$$\begin{aligned} \Delta_f |Ric|^2 &\geq 2|\nabla Ric|^2 + 2|Ric|^2 - c|Rm||Ric|^2 \\ &\geq 2|\nabla Ric|^2 + 2|Ric| - c\frac{|\nabla Ric|}{\sqrt{f}}|Ric|^2 - c|Ric|^3. \end{aligned}$$

and direct computation gives

$$\begin{aligned} \Delta_f u &= R^{-a}\Delta_f(|Ric|^2) + |Ric|^2 \Delta_f R^{-a} + 2\langle \nabla R^{-a}, \nabla |Ric|^2 \rangle, \\ R^{-a}\Delta_f(|Ric|^2) &\geq 2|\nabla Ric|^2 R^{-a} + 2|Ric|R^{-a} \\ &\quad -c\frac{|\nabla Ric|}{\sqrt{f}}|Ric|^2 R^{-a} - c|Ric|^3 R^{-a}, \\ |Ric|^2 \Delta_f R^{-a} &= |Ric|^2(-aR^{-a} + 2a|Ric|^2 R^{-a-1} + a(a+1)|\nabla R|^2 R^{-a-2}), \\ 2\langle \nabla R^{-a}, \nabla |Ric|^2 \rangle &\geq -4a|\nabla Ric||\nabla R|R^{-a-1}|Ric| \\ &\geq -a(a+1)|\nabla R|^2 R^{-a-2}|Ric|^2 - \frac{4a}{a+1}|\nabla Ric|^2 R^{-a}, \\ -c\frac{|\nabla Ric|}{\sqrt{f}}|Ric|^2 R^{-a} &\geq -\frac{2(1-a)}{1+a}|\nabla Ric|^2 R^{-a} - \frac{1+a}{8(1-a)}\frac{c^2}{f}|Ric|^4 R^{-a} \\ &\geq (-2 + \frac{4a}{1+a})|\nabla Ric|^2 R^{-a} - \frac{cR}{f}|Ric|^4 R^{-a-1}. \end{aligned}$$

Conbined those formalas, we get

$$\Delta_f u \geq (2a - \frac{cR}{(1-a)f})u^2 R^{-a} - cu^{3/2} R^{a/2}$$

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on $M \setminus D(\gamma_0)$. Choosing a constant $\gamma > 1$ abitary and let $a = 1 - \frac{1}{4\gamma}$, we can compute that on $M \setminus D(8\gamma cA)$

$$2a - \frac{cR}{(1-a)f} \geq 2 - \frac{1}{2\gamma} - \frac{cA}{(1-a)8cA\gamma} \geq \frac{3}{2} - \frac{1}{2} \geq 1 \ \ on\, M \setminus D(8\gamma cA).$$

Hence

$$\begin{aligned}\Delta_f u &\geq (2a - \frac{cR}{(1-a)f})u^2 R^{-a} - cu^{3/2}R^{a/2} \\ &\geq u^2 R^{-a} - cA^{a/2}u^{3/2} \\ &\geq cf(a)u^2 - cAu^{3/2} \\ &\geq cu^2 - cu^{3/2}.\end{aligned}$$

In line three, we use theorm 2.3. Now by theorem 3.1 we get

$$\sup_{M\setminus D(16\gamma cA)} u = \sup_{M\setminus D(16\gamma cA)} |Ric|^2 R^{\frac{1}{4\gamma}-1} \leq C.$$

Note that C is independent of A. Hence on $\Sigma(16\gamma cA)$, we have

$$|Ric|^2 R^{-1} \leq (\sup_{(M\setminus D(16\gamma cA)} |Ric|^2 R^{\frac{1}{4\gamma}-1})R^{-\frac{1}{4\gamma}} \leq CR^{-\frac{1}{4\gamma}} \leq (\frac{C}{\gamma})^{-\frac{1}{4\gamma}} \leq C.$$

In the last line we use $R \geq (\frac{c}{f}) \geq (\frac{c}{\gamma})$ by theorem 2.3. □

Theorem 4.1. *Let (M, g, f) be a four-dimensional shrinking gradient Ricci soliton with bounded scalar curvature $R \leq A$. Then there exists $\gamma_1 > 0$ depending only on A so that for any $k \in \mathbb{N}$*

$$\sup_{M\setminus D(\gamma_1)} (|Rm| + |\nabla^k Rm|) \leq C_k,$$

where $C_k > 0$ are constants depending on A only.

Proof. Let S be a tensor with $0 < |S|$, then one can compute

$$\begin{aligned}\Delta_f |S| &= \Delta_f \sqrt{|S|^2} = \frac{1}{2}\frac{1}{|S|}\Delta_f |S|^2 - \frac{1}{4}\frac{1}{|S|^3}|\nabla |S|^2|^2 \\ &= \frac{\langle S, \Delta_f S\rangle}{|S|} + \frac{|\nabla S|^2}{|S|} - \frac{|S\nabla S|^2}{|S|^3} \\ &\geq \frac{\langle S, \Delta_f S\rangle}{|S|} \geq |\Delta_f S|.\end{aligned}$$

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Now we proof the theorem by induction on k.

Step 1. $\sup_{M\backslash D(\gamma_1)} |Rm| < C$

We compute

$$\begin{aligned}
\Delta_f |Rm| &\geq |Rm| - c|Rm|^2 \\
&\geq 2|Rm|^2 - c|Rm|^2 \\
&\geq 2|Rm|^2 - c|\nabla Ric|^2 - c|Ric|^2 \\
&\geq 2|Rm|^2 - c|\nabla Ric|^2 - c, \\
\Delta_f |Ric|^2 &\geq 2|\nabla Ric|^2 - c|\nabla Ric| - c.
\end{aligned}$$

Conmbining above formulas, we get

$$\Delta_f(|Rm| + c|Ric|^2) \geq 2|Rm|^2 - c \geq (|Rm| + c|Ric|^2)^2 - c.$$

By theorem 3.1, we get the desired results.

Step 2.

Assume that $|\nabla^n Rm|$ is bounded for all $n < k-1$, we use prop 2.2 and compute

$$\begin{aligned}
\Delta_f |\nabla^k Rm| &\geq \frac{\langle \nabla^k Rm, \Delta_f \nabla^k Rm\rangle}{|\nabla^k Rm|} \\
&\geq \frac{\langle \nabla^k Rm, \nabla^k \Delta_f Rm + \frac{k}{2}\nabla^k Rm\rangle}{|\nabla^k Rm|} \\
&\quad + \frac{\langle Rm * \nabla^k Rm + \nabla^a Rm * \nabla^b Rm\rangle}{|\nabla^k Rm|} \\
&\geq \frac{\langle \nabla^k Rm, \nabla^k (Rm + Rm * Rm) + \frac{k}{2}\nabla^k Rm\rangle}{|\nabla^k Rm|} \\
&\quad + \frac{\langle Rm * \nabla^k Rm + \nabla^a Rm * \nabla^b Rm\rangle}{|\nabla^k Rm|} \\
&\geq -c|\nabla^k Rm| - c,
\end{aligned}$$

where $a, b \in 1, 2, ..., k-1$ and $a + b = k$. Also,

$$\begin{aligned}
\Delta_f |\nabla^{k-1} Rm|^2 &= 2|\nabla^k Rm|^2 + 2\langle \nabla^{k-1} Rm, \Delta_f \nabla^{k-1} Rm\rangle \\
&\geq 2|\nabla^k Rm|^2 + 2\langle \nabla^{k-1} Rm, \nabla^{k-1} \Delta_f Rm\rangle \\
&\quad + 2\langle \nabla^{k-1} Rm, \frac{k-1}{2}\nabla^{k-1} Rm + \nabla^a Rm * \nabla^b Rm\rangle \\
&\geq 2|\nabla^k Rm|^2 - c,
\end{aligned}$$

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where $a, b \in 0, 1, 2, ..., k-1$ and $a + b = k - 1$. Now we get

$$\Delta_f(|\nabla^k Rm| + |\nabla^{k-1} Rm|^2) \geq (|\nabla^k Rm| + |\nabla^{k-1} Rm|^2)^2 - c.$$

Applying theorem 3.1 again, the result fallows. □

5 Curvature Bounded by Scalar curvature

In this section we will show that

$$\sup_M \frac{|Rm|}{R} \leq C.$$

Lemma 5.1. *Let (M, g, f) be a four-dimensional shrinking gradient Ricci soliton with bounded scalar curvature $R \leq A$. Then there exists $C > 0$ such that*

$$\sup_M \frac{|Rm|^2}{R} \leq C.$$

Proof. According to lemma 4.1

$$\begin{aligned} |Rm|^2 &\leq c(\frac{|\nabla Ric|^2}{f} + |Ric|^2) \\ &\leq c(\frac{1}{f} + R) \\ &\leq cR, \end{aligned}$$

where we use $R \geq 1/f$ in line three. □

Lemma 5.2. *Let (M, g, f) be a four-dimensional shrinking gradient Ricci soliton with bounded scalar curvature $R \leq A$. Then there exists $c > 0$ such that*

$$\sup_M \frac{|\nabla Rm|^2}{R} \leq C.$$

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Proof. Let $u := |\nabla Rm|^2 R^{-1}$, we compute that

$$\begin{aligned}
\Delta_f u &= R^{-1}(\Delta_f |\nabla Rm|^2) + (\Delta_f R^{-1})|\nabla Rm|^2 \\
&\quad -4|\nabla R||\nabla^2 Rm|R^{-2}|\nabla Rm|, \\
R^{-1}(\Delta_f |\nabla Rm|^2) &= 2R^{-1}\langle \nabla Rm, \Delta_f \nabla Rm\rangle \\
&\quad +2R^{-1}|\nabla^2 Rm| \\
&\geq R^{-1}(2\langle \nabla Rm, \nabla \Delta_f Rm + \frac{1}{2}\nabla Rm + Rm * \nabla Rm\rangle) \\
&\quad +2R^{-1}|\nabla^2 Rm| \\
&\geq R^{-1}(3|\nabla Rm| + 2|\nabla^2 Rm| - c|Rm||\nabla Rm|), \\
(\Delta_f R^{-1})|\nabla Rm|^2 &= |\nabla Rm|^2(-R^{-1} + 2|Ric|^2 R^{-2} + 2|\nabla R|^2 R^{-3}, \\
-4|\nabla R||\nabla^2 Rm|R^{-2}|\nabla Rm| &\geq -2|\nabla^2 Rm|^2 R^{-1} - 2|\nabla R|^R R^{-3}|\nabla Rm|^2, \\
-c|Rm||\nabla Rm|^2 R^{-1} &\geq -c|\nabla Rm|R^{-1/2} \\
&\geq |\nabla Rm|^2 R^{-1} - c.
\end{aligned}$$

Hence we get

$$\Delta_f u \geq u - c.$$

Note that $u = |\nabla Rm|^2 R^{-1} \leq CR^{-1} \leq cf$, hence by theorem 3.1, we prove this lemma.

□

Lemma 5.3. *Let (M, g, f) be a four-dimensional shrinking gradient Ricci soliton with bounded scalar curvature $R \leq A$. Then there exists $C > 0$ such that*

$$|\nabla \ln R|^2 \leq C \ln(f + 2).$$

Proof. Let $h := (1/\epsilon)R^\epsilon$ with $\epsilon > 0$ Then we can compute

$$\Delta_f h = \epsilon h - 2\epsilon|Ric|^2 R^{-1} h + (\epsilon - 1)R^{\epsilon-2}|\nabla R|^2.$$

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If we define $\sigma := |\nabla h|^2 = R^{2\epsilon-2}|\nabla R|$, then we get

$$\begin{aligned}
\frac{1}{2}\Delta_f \sigma &= |Hess(h)|^2 + \langle \nabla h, \nabla(\Delta_f h)\rangle + Ric_f(\nabla h, \nabla f) \\
&\geq \langle \nabla h, \nabla(\Delta_f h)\rangle \\
&\geq (\epsilon-1)\langle \nabla h, \nabla R^{\epsilon-2}|\nabla R|^2\rangle - 2\epsilon\langle \nabla h, \nabla |Ric|^2 R^{-1} h\rangle, \\
(\epsilon-1)\langle \nabla h, \nabla R^{\epsilon-2}|\nabla R|^2\rangle &= (\epsilon-1)\langle \nabla h, \nabla(R^{-\epsilon}\sigma)\rangle \\
&= -(\epsilon-1)\epsilon|\nabla h|^2 R^{-2\epsilon}\sigma + (\epsilon-1)R^{-\epsilon}\langle \nabla h, \nabla \sigma\rangle, \\
-2\epsilon\langle \nabla h, \nabla |Ric|^2 R^{-1} h\rangle &\geq -4\epsilon|\nabla Ric||Ric|h|\nabla h|R^{-1} - 2\epsilon|Ric|^2 R^{-1}|\nabla h|^2 \\
&\geq -4\frac{|\nabla Ric|}{\sqrt{R}}\frac{|Ric|}{\sqrt{R}}R^{\epsilon}\sqrt{\sigma} - c\epsilon\sigma \\
&\geq -c - c\sigma.
\end{aligned}$$

Note that we use lemma 5.1 and 5.2 in last line. Combined the above formulas, we get

$$\frac{1}{2}\Delta_f \sigma \geq -(\epsilon-1)\epsilon R^{-2\epsilon}\sigma^2 + (\epsilon-1)R^{-\epsilon}\langle \nabla h, \nabla \sigma\rangle - c - c\sigma.$$

Let $\phi : \mathbb{R} \to \mathbb{R}$ be a smooth non-negtive function so that $\phi = 1$ on $[0, \gamma]$ and $\phi = 0$ for $t > 2\gamma$, and we can choose ϕ such that

$$t^2(|\phi'|^2(t) + |\phi''|(t)) \leq c.$$

Note that the choosen of c is independent of γ and it fallows that

$$|\nabla\phi(f)|^2 \leq \frac{c}{\gamma}, |\Delta_f \phi(f)| \leq c \;\; on\, M \setminus D(\gamma_0)$$

for a constant γ_0, same as in lemma 4.1. Now let $G := \phi^2\sigma$, we compute

$$\begin{aligned}
\frac{1}{2}\phi^2\Delta_f G \;\; \geq \;\; & (1-\epsilon)\epsilon R^{-2\epsilon}G^2 + (1-\epsilon)R^{-\epsilon}\langle \nabla h, \nabla G\rangle \\
& -(1-\epsilon)R^{-\epsilon}\langle \nabla h, \nabla \phi^2\rangle G - cG - c + \langle \nabla\phi^2, \nabla G\rangle.
\end{aligned}$$

At the maximun point of G we have

$$\epsilon G^2 \leq cG^{3/2}|\nabla\phi|R^{\epsilon} + cG + c \leq \frac{c}{\sqrt{\gamma}}G^{3/2} + cG + c.$$

Choosing $\epsilon := (\ln\gamma)^{-1}$, one founds that

$$(\epsilon G) \leq \frac{c\sqrt{\ln\gamma}}{\sqrt{\gamma}}\sqrt{(\epsilon G)} + c + \frac{c}{\ln\gamma}.$$

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Hence we get $\epsilon G(q) \leq C$ on the maximun point. Note that $R^\epsilon \leq c$, we have

$$\sup_{\Sigma(R)/} |\nabla R|^2 = (\sup_{\Sigma(R)/} \sigma) R^{2\epsilon} = 4(\sup_{\Sigma(R)/} G) R^{2\epsilon} \leq cG(q) \leq \frac{c}{\epsilon} \leq c ln R$$

and the result fallows. □

Theorem 5.1. *Let (M, g, f) be a four-dimensional shrinking gradient Ricci soliton with bounded scalar curvature $R \leq A$. Then there exists C such that*

$$\sup_M \frac{|Rm|}{R} \leq C.$$

Proof. First we note that by lemma 5.2 we have

$$|Rm| \leq c(\frac{|\nabla Ric|}{\sqrt{f}} + |Ric|) \leq c(S + |Ric|) \leq c|Ric|.$$

Hence let $u := |Ric|^2 R^{-2}$, we can compute that

$$\begin{aligned}
\Delta_f u &= R^{-2}\Delta(|Ric|^2) + |Ric|^2 \Delta_f(R^{-2}) + 2\langle \nabla R^{-2}, \nabla |Ric|^2 \rangle \\
&\geq 2|\nabla Ric|^2 R^{-2} + 2|Ric|^2 R^{-2} \\
&\quad -c|Ric|^3 R^{-2} + 2\langle \nabla R^{-2}, \nabla |Ric|^2 \rangle \\
&\quad +|Ric|^2(-2R^{-2} + 4|Ric|^2 R^{-3} + 6|\nabla R|^2 R^{-4}), \\
2\langle \nabla R^{-2}, \nabla |Ric|^2 \rangle &= R^2 \langle \nabla R^{-2}, \nabla(|Ric|^2 R^{-2}) \rangle - |\nabla R^{-2}|^2 ||Ric|^2 R^2 \\
&\quad +\langle \nabla R^{-2}, \nabla |Ric|^2 \rangle \\
&\geq -2\langle \nabla \ln R, \nabla(|Ric|^2 R^{-2}) \rangle - 6|\nabla R|^2 R^{-4} |Ric|^2 - 2|\nabla Ric|^2, \\
-c|Ric|^3 R^{-2} &\geq -|Ric|^4 R^{-3} - c|Ric|^2 R^{-1}.
\end{aligned}$$

Now we have

$$\Delta_f u \geq 3u^2 R - cuR - 2\langle \nabla u, \nabla \ln R \rangle.$$

It is easy to check that $\frac{|\nabla \ln R|}{\sqrt{f}}$ converges to zero at infinity by lemma 5.3. Using theorem 3.1, we get

$$\sup_M \frac{|Ric|}{R} \leq C.$$

and by $|Rm| \leq c|Ric|$, the result fallows. □

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Theorem 5.2. *Let (M, g, f) be a four-dimensional shrinking gradient Ricci soliton with bounded scalar curvature $R \leq A$. Then there exists C such that*

$$\sup_M \frac{|\nabla^k Rm|}{R} \leq C$$

forall $k \geq 0$.

Proof. First by prop 2.2, It is not hard to find that

$$\Delta_f \nabla^k Rm = (1 + \frac{k}{2})\nabla^k Rm + Rm * \nabla^k Rm + \nabla^a Rm * \nabla^b Rm,$$

where $a, b \in 1, 2, ..., k-1$ and $a + b = k$. By induction on k, we assume

$$\sup_M \frac{|\nabla^m Rm|}{R} \leq C$$

for $m \in 1, 2, ..., k-1$, and by theorem 5.1, we have

$$\begin{aligned} \Delta_f |\nabla^k Rm|^2 &\geq 2|\nabla^k Rm|^2 + (2+k)|\nabla^k Rm|^2 - c|Rm||\nabla^k Rm|^2 - c|\nabla^a Rm||\nabla^b Rm| \\ &\geq 2|\nabla^k Rm|^2 + (2+k)|\nabla^k Rm|^2 - cR|\nabla^k Rm|^2 - cR^2. \end{aligned}$$

Now let $u := |\nabla^k Rm|^2 R^{-2}$, we compute

$$\begin{aligned} \Delta_f u &= R^{-2}\Delta_f(|\nabla^k Rm|^2) + |\nabla^k Rm|^2 \Delta_f(R^{-2}) \\ &\quad +2\langle \nabla R^{-2}, \nabla|\nabla^k Rm|^2\rangle \\ &\geq 2|\nabla^{k+1} Rm|^2 R^{-2} + (2+k)|\nabla^k Rm|^2 R^{-2} \\ &\quad -c|\nabla^k Rm|^2 R^{-1} - c + 2\langle \nabla R^{-2}, \nabla|\nabla^k Rm|^2\rangle \\ &\quad +|\nabla^k Rm|^2(-2R^{-2} + 4|Ric|^2 R^{-3} + 6|\nabla R|^2 R^{-4}), \\ 2\langle \nabla R^{-2}, \nabla|\nabla^k Rm|^2\rangle &= R^2\langle \nabla R^{-2}, \nabla(|\nabla^k Rm|^2 R^{-2})\rangle \\ &\quad -|\nabla R^{-2}|^2||\nabla^k Rm|^2 R^2 + \langle \nabla R^{-2}, \nabla|\nabla^k Rm|^2\rangle \\ &\geq -2\langle \nabla \ln R, \nabla(|Ric|^2 R^{-2})\rangle - 6|\nabla^k Rm|^2 R^{-4}|Ric|^2 \\ &\quad -2|\nabla^{k+1} Rm|^2, \\ 4|Ric|^2|\nabla^k Rm|^2 R^{-3} &\geq -c|\nabla^k Rm|^2 R^{-1}. \end{aligned}$$

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where in the last line we apply theorem 5.1, Now we get

$$\Delta_f u \geq ku - cu^{\frac{1}{2}}|\nabla Rm| - c - 2\langle \nabla u, \nabla \ln R\rangle.$$

By theorem 4.1, $|\nabla^k Rm| < C$, we have

$$\Delta_f u \geq ku^2 - cu^{\frac{1}{2}} - c - 2\langle \nabla u, \nabla \ln R\rangle\ lequ^2 - c - 2\langle \nabla u, \nabla \ln R\rangle,$$

where we estimate $-cu^{\frac{1}{2}} \leq -(k-1)u^2 - \frac{c^2}{4k-4}$. By theorem 3.2, the result fallows. □

6 Curevature lower bound

Theorem 6.1. *Let (M, g, f) be a four-dimensional shrinking gradient Ricci soliton with bounded scalar curvature $R \leq A$. Then there exists a constant c such that*

$$Rm \geq -(\frac{c}{\log(r+1)})^{\frac{1}{4}}$$

Proof. Choose an orthonormal basis $\{e_i\}$ with $\{e_1, e_2, e_3\}$ locate on $T\Sigma(t)$, where t is big enough such that Σ forms a submanifold, and $e_4 = \frac{\nabla f}{|\nabla f|}$. In the proof of lemma 4.1, we shown that

$$R_{1212} = -R_{11} - R_{22} + \frac{R}{2} + R_{3434}, \quad R_{1223} = R_{13} + R_{1434},$$
$$R_{2323} = -R_{22} - R_{33} + \frac{R}{2} + R_{1414}, \quad R_{2331} = R_{21} + R_{2414},$$
$$R_{3131} = -R_{11} - R_{33} + \frac{R}{2} + R_{2424}, \quad R_{1231} = R_{23} + R_{2434}.$$

Choose $\{e_1, e_2, e_3\}$ which diagonalize the Ricci curvarure restrict on $T\Sigma$ and satisfied $R_{11} \leq R_{22} \leq R_{33}$. define

$$R_{11} = \lambda_1,\ R_{22} = \lambda_2,\ R_{33} = \lambda_3,$$

$$\nu = \lambda_1 + \lambda_2 - \lambda_3,$$
$$\lambda = \lambda_1 + \lambda_3 - \lambda_2,$$
$$\mu = \lambda_2 + \lambda_3 - \lambda_1.$$

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Easy to see that $\nu \le \lambda \le \mu$, and we can find

$$Rm \ge \frac{1}{2}\nu - c|R_{ijk4}| \ge \frac{1}{2}\nu - c\frac{|\nabla Ric|}{\sqrt{f}} \ge \frac{1}{2}\nu - cRf^{-1/2}.$$

In order to prove the theorem, it is sufficient to show

$$\nu \ge -(\frac{c}{\log(r+1)})^{\frac{1}{4}}.$$

By prop 2.2, we have $\Delta_f R_{ab} = R_{ab} - 2R_{iabj}R_{ij}$, and compute

$$\begin{aligned}\Delta_f \lambda_a &= R_{aa} - 2R_{iaaj}R_{ij} + R_{4aa4}R_{44} \\ &= R_{aa} - 3\overline{R}R_{aa} + \overline{R} + 4R_{ai}R_{ai} - 2|\overline{Ric}|^2 + O(S^2 f^{-1/2}).\end{aligned}$$

Where $\overline{R} := R_{11} + R_{22} + R_{33}$, $|\overline{Ric}|^2 := R_{11}^2 + R_{22}^2 + R_{33}^2$. Now one can compute that

$$\Delta_f \nu \le \nu - \nu^2 - \lambda\mu + cRf^{-1/2}.$$

Define $F = f - 2\ln R$ and let $u := \frac{\nu}{R}$ then compute

$$\begin{aligned}\Delta_f u &= (\Delta_f \nu) + \nu(\Delta_f R^(-1) + 2\langle \nabla\nu, \nabla R^{-1}\rangle + 2\langle \ln R, \nabla(\nu R^{-1})\rangle \\ &\le (\nu - \nu^2 - \lambda\mu + cRf^{-1/2})R^{-1} + 2\langle \ln R, \nabla(\nu R^{-1})\rangle \\ &\quad + \nu(-R^{-1} + 2|Ric|^2 R^{-2} + 2|\nabla R|^2 R^{-3}) \\ &\quad + 2\langle \nabla(\nu R^{-1}), \nabla R^{-1}\rangle R + 2\langle \nabla R, \nabla R^{-1}\rangle(\nu R^{-1}) \\ &= -R^{-2}((\nu^2 + \lambda\mu)R - 2|Ric|^2\nu) + cf^{-1/2} \\ &\le -R^{-2}(\lambda^2(\mu - \nu) + \mu^2(\lambda - \nu)) + cf^{-1/2}.\end{aligned}$$

Where in the last line we use

$$|Ric|^2 \le |\overline{Ric}|^2 + cR^2 f^{-1/2} = (\lambda^2 + \mu^2 + \nu^2 - \lambda\mu + \lambda\nu + mu\nu) + cR^2 f^{-1/2}.$$

By Theorem 5.1, we know there is constants c_0 such that $u > -c_0$. Now we define

$$w := u + kf^{-\epsilon} + \epsilon R^{-1}$$

where $\epsilon = \frac{1}{\sqrt{\gamma_1}} = \frac{1}{\sqrt{\ln \gamma}}$, $k := c_0(\gamma_1)^{-\epsilon}$. We can choose k such that $w > 0$ on $M \setminus D(\gamma_1)$ and compute that

$$\Delta_F w \le -R^{-2}(\lambda^2(\mu - \nu) + \mu^2(\lambda - \nu)) + 2\epsilon k f^{-\epsilon} - \epsilon R^{-1} + c\epsilon.$$

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By choose $\psi := 1 - \frac{t}{\gamma}$ and let $G := \psi^2 w$, we can get that on the minimum point q of G

$$0 \le -R^{-2}(\lambda^2(\mu-\nu)+\mu^2(\lambda-\nu))\psi^4 + (2\epsilon k f^{-\epsilon} - \epsilon R^{-1} + c\epsilon)\psi^4 + (\psi\frac{f}{\gamma} - 6|\nabla\psi|)G.$$

Now we concider two cases: Case 1. $(\psi\frac{f}{\gamma} - 6|\nabla\psi|)G \le 0.$

Then

$$R^{-2}(\lambda^2(\mu-\nu)+\mu^2(\lambda-\nu)) \le (2\epsilon k f^{-\epsilon} - \epsilon R^{-1} + c\epsilon),$$

and

$$\begin{aligned} \epsilon R^{-1} &\le 2k\epsilon f^{-\epsilon} + c\epsilon \\ &\le 2k\epsilon(\gamma_1)^{-\epsilon} + c\epsilon \\ &= (c_0 + c)\epsilon. \end{aligned}$$

Hence there exist c_1 such that $S(q) \ge \frac{1}{c_1}$. Now we have

$$\lambda^2(\mu-\nu)+\mu^2(\lambda-\nu) \le c\epsilon.$$

In addition, we know $\overline{R} = \lambda + \mu + \nu \ge \frac{1}{2c_1}$, which implies $\mu \ge \frac{1}{4c_1}$. By $\lambda^2(\mu-\nu) \le c\epsilon$ and $\nu(q) \le 0$, we get $|\lambda| \le c_2\sqrt{\epsilon}$. And then $\lambda^2(\mu-\nu) \le c\epsilon$ implies

$$-\nu - c_2\sqrt{\epsilon} \le \lambda - \nu \le \frac{c\epsilon}{\mu} \le c\epsilon.$$

This proves $G(q) \ge -c\sqrt{\epsilon}$ and

$$\inf_{D(\gamma/2)\setminus D(\gamma_1)} w \ge -c\sqrt{\epsilon}$$

Case 2. $(\psi\frac{f}{\gamma} - 6|\nabla\psi|)G \ge 0.$

Since $G(q) \le 0$. We would get $\psi(q) \le 6/\gamma$ and

$$\inf_{D(\gamma/2)\setminus D(\gamma_1)} w \ge -\frac{c}{\gamma^2}.$$

To sumurize,

$$\inf_{D(\gamma/2)\setminus D(\gamma_1)} \nu \ge -c(\frac{\gamma_1}{f})^{\epsilon} - c\epsilon.$$

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Plug $\epsilon = \frac{1}{\sqrt{\gamma_1}}$ into it, and take γ abitary, we get

$$\nu \geq -\left(\frac{c}{\log(r+1)}\right)^{\frac{1}{4}}.$$

$\square$

References

[1] O. Munteanu and J. P. Wang , *Geometry of shrinking Ricci solitons*. Compositio Mathematica, 151, 2273-2300 (2015)

[2] H. D. Cao and D. Zhou, *On complete gradient shrinking Ricci solitons*, J. Diff. Geom. 85, no. 2, 175-186 (2010)

[3] B. L. Chen, *Stron uniqueness of the Ricci flow*, J. Diff. Geom. 82, no. 2,362-382 (2009)

[4] B. Chow, R. Lu and B. Yang, *A lower bound for the scalar curvature of noncompact nonflat Ricci shrinkers*, Comptes Rendus Mathematique. 349, no. 23-24, 1265-1267 (2011)

[5] J. Enders, R. Müller, P. Topping, *On Type-I singularities in Ricci Flow*, Comm. Anal. Geom. 19, , no. 5, 905-922 (2011)

[6] P. Petersen and W. Wylie, *On the classification of gradient Ricci solitons*, Geom. Topol. 14, ,2277-2300 (2010)

doi:10.6342/NTU201601353