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線性差分方程的可解性

Solvability of Singular Linear Difference Equations

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國立臺灣大學碩士學位論文
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Solvability of Singular Linear Difference Equations

本論文係林育任君（R99221009）在國立臺灣大學數學系、所完成之碩士學位論文，於民國 101 年 6 月 22 日承下列考試委員審查通過及口試及格，特此證明

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（是否須簽章依各院系所規定）

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摘要

這篇論文最主要是在探討關於線性差分方程的可解性。主要是以幾何的觀點去探討關於(E, A, B)-系統的解的性質。我們先以較簡單的(E, A)-系統入手，並且嘗試著利用幾何的觀點去探討出其解的性質。並且希望可以將求解的方式，以與所選取基底無關的方法來獲得相關結論。

而(E, A)-系統為(E, A, B)-系統的特例。因此之後可利用之前的結論，再進一步地研究關於(E, A, B)-系統解的特性。而在最後，也得以完整的描述解空間。

關鍵字：可解性、線性差分方程、(E, A)-系統、(E, A, B)-系統、幾何控制

Abstract

In this thesis, we focus on the solvability of singular linear difference equations. We use the geometric viewpoint to survey the properties about the solutions of (E, A, B)-system. First, we consider the simple system—(E, A)-system. We try to use the geometric technique to solve the properties about the solutions of (E, A)-system. And we hope that we can solve it by the way which is independent of the choice of the basis.

And (E, A)-system is a special case of the (E, A, B)-system. So, we can use the conclusions which we got before to solve the solution of the (E, A, B)-system. Finally, we have described the solution space of (E, A, B)-system.

Key words: solvability, singular linear difference equations,
(E, A)-system , (E, A, B)-system, geometric control

目　錄

Chapter 1

Introduction

In this thesis, we want to talk about the discrete-time singular systems. The fundamental discrete-time singular systems is described by the following equation:

$$Ex_{k+1} = Ax_k + Bu_k, \quad \text{for all } k \in \mathbb{N}.$$

We usually use the triple (E, A, B) to represent the equation. This equation has been studied in many years. In the literature, the matrices operation was usually used to solve the equation with nonsingular E, but the results are not easy to extend to the general case for singular E. The purpose of the thesis is to study the solvability of the equation for singular E case.

At the begining of this thesis, we will state some basic definitions and properties. Those will give us some tools at the later chapter. Later, we will start from the solvability of the ordered pair (E, A), and then extend it to (E, A, B) systems.

Chapter 2

Preliminaries

Notation. We denote the set of non-negative integers by $\mathbb{N}$ and the set of positive integers by $\mathbb{N}^*$. Set $J_k := \{0, 1, 2, \cdots, k\}$ and $J_k^* := \{1, 2, \cdots, k\}$.

Let $\mathcal{X}$ be an n-dimensional vector space over an algebraically closed field $\mathbb{F}$, and let A and E be linear transformations on $\mathcal{X}$. Assume that $\mathrm{rank}E = r \leq n$. The ordered pair (also called a *pencil*) (E, A) is said to be *regular* if there exists a scalar $\lambda \in \mathbb{F}$ such that $\lambda E - A$ is non-singular. Clearly, (E, A) is regular if E is non-singular. Also, if (E, A) is regular, we have that $\mathrm{im}E + \mathrm{im}A = \mathcal{X}$ [11].

Suppose that (E, A) is regular. A point $(u : v)$ in the projective line $\mathbb{P}^1(\mathbb{F})$ is called an *eigenvalue* of (E, A) if $uE + vA$ is singular; any non-zero vector $x \in \ker(uE + vA)$ is called an *eigenvector* of (E, A) corresponding to $(u : v)$. By Bezout's theorem, a regular pencil has at most n eigenvalues. The set of all eigenvalues of (E, A), denoted by $\sigma(E, A)$, is called the *spectrum* of (E, A).

Suppose that (E, A) is regular. An eigenvalue $(u : v)$ of (E, A) is said to be infinite if $v = 0$; otherwise $(u : v)$ is said to be finite. By convention, we shall usually say that $\lambda := -u/v$ is a finite eigenvalue of (E, A) if $(u : v)$ is a finite eigenvalue of (E, A). Sometimes we use the notation $\sigma_f(E, A)$ for the set of all finite eigenvalues of (E, A). We shall also say that (E, A) has an infinite eigenvalue or has an eigenvalue at infinity (usually denoted $\lambda = \infty$) if $(1 : 0)$ is an eigenvalue of (E, A).

Convention. Given a linear transformation $E : V \to W$, we always denote $E^{-1}(U)$

(resp. $E^{-1}(w)$) to be the preimage of a subspace $U \subset W$ (resp. a vector $w \in W$) in this article.

Let $\mathcal{E}_1 := \ker E = E^{-1}(0)$, and $\mathcal{E}_{i+1} := E^{-1}A\mathcal{E}_i$ for all $i \in \mathbb{N}$. For convenience, we set $\mathcal{E}_0 := 0$. It is easy to show that $\mathcal{E}_{i-1} \subset \mathcal{E}_i$ and $E\mathcal{E}_i \subset A\mathcal{E}_i$ for all $i \in \mathbb{N}^*$. Moreover, if $\mathcal{E}_i = \mathcal{E}_{i+1}$ for some $i \in \mathbb{N}$, then $\mathcal{E}_m = \mathcal{E}_i$ for all $m \geq i$ [11].

We let $\mathcal{E}_\infty := \bigcup_{i=0}^{\infty} \mathcal{E}_i$. Note that $\mathcal{E}_\infty = \mathcal{E}_j$ for some j because our vector space is always finite dimensional and indeed the equality holds if $\mathcal{E}_j = \mathcal{E}_{j+1}$.

The index of (E, A), denoted by $\text{ind}(E, A)$, is defined as the smallest non-negative integer j such that $\mathcal{E}_j = \mathcal{E}_{j+1}$. By definition, $\text{ind}(E, A) = 0$ if and only if E is non-singular. Let $\kappa := \text{ind}(E, A)$. Then (E, A) is regular if and only if $\ker A \cap \mathcal{E}_\kappa = 0$ [10].

By symmetry, let $\mathcal{A}_0 := 0$, and $\mathcal{A}_i := A^{-1}E\mathcal{A}_{i-1}$ for all $i \in \mathbb{N}^*$, where A^{-1} denotes the preimage under A. We have $\mathcal{A}_{i-1} \subset \mathcal{A}_i$ and $A\mathcal{A}_i \subset E\mathcal{A}_i$ for all $i \in \mathbb{N}^*$. Moreover, if $\mathcal{A}_l = \mathcal{A}_{l+1}$ for some $l \in \mathbb{N}$, then $\mathcal{A}_m = \mathcal{A}_l$ for all $m \geq l$.

Let $\mathcal{A}_\infty := \bigcup_{i=0}^{\infty} \mathcal{A}_i$. Then $\mathcal{A}_\infty = \mathcal{A}_l$ for some l. The index of (A, E), denoted by $\text{ind}(A, E)$, is defined as $\text{ind}(A, E) := \min\{l | \mathcal{A}_l = \mathcal{A}_{l+1}\}$. Let $\iota := \text{ind}(A, E)$. Then (E, A) is regular if and only if $\ker E \cap \mathcal{A}_\iota = 0$. This follows from the fact that (E, A) is regular if and only if (A, E) is regular.

Suppose that (E, A) is regular. Let $\lambda \in \sigma_f(E, A)$. Let $\mathcal{V}_{\lambda,0} := 0$, $\mathcal{V}_{\lambda,1} := \ker(\lambda E - A)$, and inductively, let $\mathcal{V}_{\lambda,k+1} := (\lambda E - A)^{-1}E\mathcal{V}_{\lambda,k}$ for all $k \in \mathbb{N}^*$. Then it can be shown by induction that $\mathcal{V}_{\lambda,k} \subset \mathcal{V}_{\lambda,k+1}$ for all $k \in \mathbb{N}$. Since $\mathcal{X}$ has finite dimension, there is a minimum index L such that $\mathcal{V}_{\lambda,L} = \mathcal{V}_{\lambda,L+i}$ for all $i \in \mathbb{N}^*$. Let $\mathcal{V}_{\lambda,\infty} := \bigcup_{k=0}^{\infty} \mathcal{V}_{\lambda,k}$. Then $\mathcal{V}_{\lambda,\infty} = \mathcal{V}_{\lambda,L}$. $\mathcal{V}_{\lambda,\infty}$ is called the *eigenspace of* (E, A) *associated with the finite eigenvalue* λ. If $\sigma_f(E, A) = \emptyset$, set $\mathcal{E}_f := 0$; otherwise, set $\mathcal{E}_f := \sum_{\lambda \in \sigma_f(E,A)} \mathcal{V}_{\lambda,\infty}$. $\mathcal{E}_f$ is called the *finite eigenspace* of (E, A). In contrast, $\mathcal{E}_\infty$ is called the *infinite eigenspace* of (E, A). It can be shown that $\mathcal{X} = \mathcal{E}_f \oplus \mathcal{E}_\infty$.

A subspace $\mathcal{V}$ of $\mathcal{X}$ is said to be a *deflating subspace* of (E, A) or simply, *deflating* for (E, A), or more simply, (E, A)-*deflating*, if there exists a subspace $\mathcal{W}$ of $\mathcal{X}$, called a *codeflating subspace* of $\mathcal{V}$, such that $E\mathcal{V} \subset \mathcal{W}$, $A\mathcal{V} \subset \mathcal{W}$, and $\dim \mathcal{V} = \dim \mathcal{W}$. When

(E, A) is regular, $\mathcal{W}$ is uniquely determined by $\mathcal{W} = E\mathcal{V} + A\mathcal{V}$.

Clearly, $\mathcal{V}$ is (I, A)-deflating if and only if $\mathcal{V}$ is A-invariant, where **I** stands for the identity transformation on $\mathcal{X}$. Thus, the concept of a deflating subspace generalizes that of an invariant subspace.

Two pencils (E', A') and (E, A) are called *strictly equivalent* if $(E', A') = (P_1EP_2, P_1AP_2)$ for some non-singular linear transformations P_1 and P_2 on $\mathcal{X}$. Clearly, two strictly equivalent pencils have the same index.

A pencil (E, A) is regular if and only if it is strictly equivalent to a pencil with a matrix pencil representation in Weierstrass canonical form:

$$(\begin{pmatrix} I_q & 0 \\ 0 & N \end{pmatrix}, \begin{pmatrix} \Lambda & 0 \\ 0 & I_{n-q} \end{pmatrix}), \tag{2.1}$$

where I_k denotes the $k \times k$ identity matrix and N is nilpotent [2, 1]. Without loss of generality, we may assume that both N and Λ are Jordan matrices. Note that the block corresponding to N is void if E is non-singular; in this case, the nilpotent index of N is defined to be zero. Moreover, it can be shown that the index of (E, A) is precisely the nilpotent index of N [11].

Give a matrix A, there exist a unique matrix X satisfying the following equations (we usually call these equations *Penrose equations*):

$$AXA = A \tag{1}$$

$$XAX = X \tag{2}$$

$$(AX)^* = AX \tag{3}$$

$$(XA)^* = XA \tag{4}$$

where A^* denote the conjugate transpose of A. The uniqueness of X generalizes the the inverse matrix of A by Moore. So, we usually call the the unique matrix X *Moore-Penrose inverse* (or $\{1, 2, 3, 4\}$-*inverse of A*) and is denoted it by A^+.

Then, we talk about another generalized inverse matrix. The *Drazin inverse* of A (or called $\{1^k, 2, 5\}$-*inverse of A*) is the unique matrix X which has the following properties:

$$A^k X A = A^k \tag{1^k}$$

$$XAX = X \tag{2}$$

$$AX = XA \tag{5}$$

And we usually denote it by A^D [14]. Notice that k will satisfy that $k \geq ind(A, I)$. These generalized inverse matrices will provide some useful result in this thesis.

Chapter 3

Solvability of (E, A)

Definition 3.1. *Let x_0 be given. We say that (E, A) is solvable for x_0 or that x_0 is a solvable state of (E, A) if there exists a sequence $(x_k)_{k=1}^{\infty}$ satisfying $Ex_{k+1} = Ax_k$ for all $k \in \mathbb{N}$. In this case, the sequence $(x_k)_{k=1}^{\infty}$ is called a fundamental sequence of (E, A) for x_0.*

Definition 3.2. *We say that a subspace $\mathcal{V}$ of $\mathcal{X}$ is a solution space of (E, A) if (E, A) is solvable for every $x_0 \in \mathcal{V}$ with a fundamental sequence $(x_k)_{k=1}^{\infty}$ lying entirely in $\mathcal{V}$.*

It is clear that the sum of finitely many solution spaces of (E, A) is also a solution space of (E, A). It follows that there exists a unique maximal solution space $\mathcal{V}^*$ of (E, A) in the sense that if $\mathcal{V}$ is any solution space of (E, A) then $\mathcal{V} \subset \mathcal{V}^*$.

By definition, we immediately have the following result.

Lemma 3.3. *$\mathcal{V}$ is a solution space of (E, A) if and only if $A\mathcal{V} \subset E\mathcal{V}$.*

Proof. Suppose that $\mathcal{V}$ is a solution space of (E, A). Then for any $x_0 \in \mathcal{V}$, there exists $x_1 \in \mathcal{V}$ such that $Ex_1 = Ax_0 \in E\mathcal{V}$. This implies that $A\mathcal{V} \subset E\mathcal{V}$. On the other hand, suppose that $A\mathcal{V} \subset E\mathcal{V}$. Let $x_0 \in \mathcal{V}$. Then there exists $x_1 \in \mathcal{V}$ such that $Ex_1 = Ax_0$. By induction, we can prove that given $x_k \in \mathcal{V}$, there exists $x_{k+1} \in \mathcal{V}$ such that $Ex_{k+1} = Ax_k$ for all $k \in \mathbb{N}$. $\square$

Proposition 3.4. *Suppose that (E, A) is regular. If $\lambda \in \sigma_f(E, A)$, then $\mathcal{V}_{\lambda,k}$ is a solution space of (E, A) for all $k \in \mathbb{N}$. In particular, $\mathcal{E}_f$ is a solution space of (E, A).*

Proof. For any $x \in \mathcal{V}_{\lambda,k}$, there exists $y \in \mathcal{V}_{\lambda,k-1}$ such that $(\lambda E - A)x = -Ey$. This implies that $Ax = \lambda Ex + Ey \in E\mathcal{V}_{\lambda,k} + E\mathcal{V}_{\lambda,k-1} = E\mathcal{V}_{\lambda,k}$. Hence $A\mathcal{V}_{\lambda,k} \subset E\mathcal{V}_{\lambda,k}$, thus proving that $\mathcal{V}_{\lambda,k}$ is a solution space of (E, A). Since (E, A) has at most n distinct eigenvalues by Bezout's theorem, $\mathcal{E}_f$ is the sum of finitely many solution spaces of (E, A). Hence $\mathcal{E}_f$ is a solution space of (E, A). □

Theorem 3.5. *Suppose that (E, A) is regular. Then $\mathcal{V}$ is a solution space of (E, A) if and only if $\mathcal{V}$ is (E, A)-deflating and $\mathcal{V} \cap \ker E = 0$.*

Proof. Suppose that $\mathcal{V}$ is a solution space of (E, A). Then $A\mathcal{V} \subset E\mathcal{V}$. Since (E, A) is regular, there exists λ such that $\lambda E - A$ is non-singular. We have $(\lambda E - A)\mathcal{V} \subset E\mathcal{V} + A\mathcal{V} \subset E\mathcal{V}$. This implies that $\dim(\lambda E - A)\mathcal{V} \leq \dim E\mathcal{V} \leq \dim \mathcal{V} = \dim(\lambda E - A)\mathcal{V}$. Hence, $\dim \mathcal{V} = \dim E\mathcal{V}$. As a result, $\mathcal{V} \cap \ker E = 0$ and $\mathcal{V}$ is (E, A)-deflating with a codeflating subspace $E\mathcal{V}$.

Since (E, A) is regular and $\mathcal{V}$ is (E, A)-deflating, the corresponding codeflating subspace is uniquely determined as $E\mathcal{V} + A\mathcal{V}$. Thus $\dim \mathcal{V} = \dim(E\mathcal{V} + A\mathcal{V})$. Since $\mathcal{V} \cap \ker E = 0$, we also have $\dim \mathcal{V} = \dim E\mathcal{V}$. Thus $\dim(E\mathcal{V} + A\mathcal{V}) = \dim E\mathcal{V}$. This implies that $A\mathcal{V} \subset E\mathcal{V}$. Hence $\mathcal{V}$ is a solution space of (E, A). □

Corollary 3.6. *Suppose that (E, A) is regular.*

1. *If $\mathcal{V}$ is a solution space of (E, A) contained in $\ker E$, then $\mathcal{V} = 0$.*

2. *If E is singular, then $\mathcal{X}$ cannot be a solution space of (E, A).*

Proof.

1. Since $A\mathcal{V} \subset E\mathcal{V} = 0$, we have $A\mathcal{V} = 0$. This implies that $\mathcal{V} \subset \ker E \cap \ker A$. Since (E, A) is regular, $\ker E \cap \ker A = 0$. Therefore, $\mathcal{V} = 0$.

2. If $\mathcal{X}$ is a solution space of (E, A), We have $A\mathcal{X} \subset E\mathcal{X}$. Since (E, A) is regular, we have $\mathcal{X} = \mathrm{im}E + \mathrm{im}A = E\mathcal{X} + A\mathcal{X} = E\mathcal{X}$. Hence $\ker E = 0$, that is, E is non-singular, which is a contradiction.

□

Theorem 3.7. *Suppose that (E, A) is regular. Then, if (E, A) is solvable for x_0, the fundamental sequence for x_0 is unique.*

Proof. Let $(x_k)_{k=1}^{\infty}$ and $(y_k)_{k=1}^{\infty}$ be fundamental sequences for x_0. For convenience we let $y_0 := x_0$. Then we have $E(y_{k+1} - x_{k+1}) = A(y_k - x_k)$ for all $k \in \mathbb{N}$. In particular, we have $E(y_1 - x_1) = 0$, and hence $y_1 - x_1 \in \ker E$. Let $\mathcal{V}$ be the subspace generated by the set $\{x_i, y_i\}_{i=0}^{\infty}$. Clearly, $\mathcal{V}$ is a solution space of (E, A) and $y_1 - x_1 \in \mathcal{V}$. Since $\mathcal{V} \cap \ker E = 0$ by Theorem 3.5, we find that $y_1 = x_1$. Inductively, we can prove that $y_k = x_k$ for all $k \in \mathbb{N}^*$. □

Corollary 3.8. *Suppose that (E, A) is regular. Then the unique fundamental sequence for $x_0 = 0$ is the zero sequence.*

Theorem 3.9. *Suppose that (E, A) is regular. Then the unique maximal solution space of (E, A) equals $\mathcal{E}_f$.*

Proof. Let $\mathcal{V}^*$ be the maximal solution space of (E, A). Let $m := \mathrm{ind}(E, A)$ and let $x_0 \in \mathcal{V}^* \cap \mathcal{E}_\infty$. Then there exists a sequence $(x_k)_{k=1}^{\infty}$ with $x_k \in \mathcal{V}^*$ for all $k \in \mathbb{N}^*$ satisfying $Ex_{k+1} = Ax_k$ for all $k \in \mathbb{N}$. Let $y_{m-1} := x_0 \in \mathcal{E}_\infty = \mathcal{E}_m$. This implies that there exist y_k, $k \in \{0, 1, 2, \cdots, m-2\}$, with $y_k \in \mathcal{E}_{k+1}$, such that $Ey_k = Ay_{k-1}$ for all $k \in \{1, 2, \cdots, m-1\}$. Let $y_{m+i-1} = x_i$ for all $i \in \mathbb{N}^*$. It is clear that $(y_k)_{k=1}^{\infty}$ is a fundamental sequence for y_0. Now let $\mathcal{V}$ be the subspace generated by the set $\{y_k\}_{k=0}^{\infty}$. Clearly, $\mathcal{V}$ is a solution space of (E, A). Hence $\mathcal{V} \cap \ker E = 0$. Since $y_0 \in \mathcal{V} \cap \ker E$, we have $y_0 = 0$. This implies that $(y_k)_{k=1}^{\infty}$ is the zero sequence. In particular, $x_0 = y_{m-1} = 0$. Hence $\mathcal{V}^* \cap \mathcal{E}_\infty = 0$. Because $\mathcal{E}_f$ is a solution space of (E, A), $\mathcal{E}_f$ is contained in $\mathcal{V}^*$. We claim that in fact $\mathcal{E}_f = \mathcal{V}^*$. Suppose, by contradiction, that $\mathcal{E}_f \subsetneqq \mathcal{V}^*$. Since $\mathcal{X} = \mathcal{E}_f \oplus \mathcal{E}_\infty$, there exists nonzero x such that $x \in \mathcal{V}^* \cap \mathcal{E}_\infty$, which is a contradiction. This completes the proof. □

Corollary 3.10. *Suppose that (E, A) is regular. Then $A^{-1}E\mathcal{E}_f = \mathcal{E}_f$.*

Proof. Since $A\mathcal{E}_f \subset E\mathcal{E}_f$, $\mathcal{E}_f \subset A^{-1}E\mathcal{E}_f$. Hence $AA^{-1}E\mathcal{E}_f \subset E\mathcal{E}_f \subset EA^{-1}E\mathcal{E}_f$. So $A^{-1}E\mathcal{E}_f$ is also a solution space of (E, A). By Theorem 3.9, $A^{-1}E\mathcal{E}_f \subset \mathcal{E}_f$. This shows that $A^{-1}E\mathcal{E}_f = \mathcal{E}_f$. □

The following theorem shows that if (E, A) is regular then the finite eigenspace of (E, A) is precisely the set of all solvable states of (E, A).

Theorem 3.11. *Suppose that (E, A) is regular. Then (E, A) is solvable for x_0 if and only if $x_0 \in \mathcal{E}_f$.*

Proof. (E, A) is solvable for every $x_0 \in \mathcal{E}_f$ since $\mathcal{E}_f$ is a solution space of (E, A). This proves the sufficiency. On the other hand, suppose that (E, A) is solvable for x_0. Let $(x_k)_{k=1}^{\infty}$ be the fundamental sequence for x_0. Let $\mathcal{V}$ be the subspace generated by the set $\{x_k\}_{k=0}^{\infty}$. Then $\mathcal{V}$ is a solution space of (E, A). Thus $x_0 \in \mathcal{V} \subset \mathcal{E}_f$. $\square$

The next theorem provides a numerical method to evaluate fundamental sequences for solvable states.

Theorem 3.12. *Let $x_0 \in \mathcal{E}_f$. Then the unique fundamental sequence $(x_k)_{k=1}^{\infty}$ for x_0 is given by*

$$x_k = V_f \Lambda^k V_f^+ x_0 \in \mathcal{E}_f, \tag{3.1}$$

for all $k \in \mathbb{N}$, where V_f is any basis matrix for $\mathcal{E}_f$, and V_f^+ denotes the Moore-Penrose inverse of V_f. In particular, $x_0 = V_f V_f^+ x_0$.

Proof. Let $(x_k)_{k=1}^{\infty}$ be the fundamental sequence for x_0. Let V_f and V_∞ be basis matrices for $\mathcal{E}_f$ and $\mathcal{E}_\infty$, respectively. By Weierstrass theorem, there exist unique matrices Λ and N such that $EV_f\Lambda = AV_f$ and $EV_\infty = AV_\infty N$, where N is nilpotent. Let $V := \begin{pmatrix} V_f & V_\infty \end{pmatrix}, W := \begin{pmatrix} EV_f & AV_\infty \end{pmatrix}$. Then,

$$W^{-1}EV = \begin{pmatrix} I & 0 \\ 0 & N \end{pmatrix}, \quad W^{-1}AV = \begin{pmatrix} \Lambda & 0 \\ 0 & I \end{pmatrix}.$$

Let $\bar{x}_k := \begin{pmatrix} \bar{x}_{1k} \\ \bar{x}_{2k} \end{pmatrix} := V^{-1}x_k$. We have

$$\begin{pmatrix} I & 0 \\ 0 & N \end{pmatrix} \begin{pmatrix} \bar{x}_{1(k+1)} \\ \bar{x}_{2(k+1)} \end{pmatrix} = \begin{pmatrix} \Lambda & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} \bar{x}_{1k} \\ \bar{x}_{2k} \end{pmatrix},$$

from which we obtain that for all $k \in \mathbb{N}$, $\bar{x}_{1k} = \Lambda^k \bar{x}_{10},\ \bar{x}_{2k} = 0$. With $x_0 = V\bar{x}_0 =$

$V_f\bar{x}_{10}$, we get $\bar{x}_{10} = V_f^+ x_0$. Hence for all $k \in \mathbb{N}$, we have $x_k = V\bar{x}_k = V_f\bar{x}_{1k} = V_f\Lambda^k\bar{x}_{10} = V_f\Lambda^k V_f^+ x_0$. $\square$

In the remaining content, we will state the properties of terminating sequence.

Definition 3.13. *A sequence $(x_k)_{k=1}^{\infty}$ is said to be* terminating *if there exists $N \in \mathbb{N}^*$ such that $x_k = 0$ for all $k \geq N$. A solution space $\mathcal{V}$ of (E, A) is said to be* terminating *if the fundamental sequences for every $x_0 \in \mathcal{V}$ are terminating.*

Proposition 3.14. *Suppose that (E, A) is regular and that $\mathcal{V}$ is an (E, A)-deflating subspace contained in $\mathcal{A}_i$ for some $i \in \mathbb{N}$. Then $\mathcal{V}$ is a terminating solution space of (E, A). More explicitly, for any $x_0 \in \mathcal{V} \subset \mathcal{A}_i$ for some $i \in \mathbb{N}$, the fundamental sequence $(x_k)_{k=1}^{\infty}$ for x_0 is terminating: $x_k = 0$ for all $k \geq i$.*

Proof. The case of $i = 0$ is trivial. If $\mathcal{V} \subset \mathcal{A}_i$ for some $i \in \mathbb{N}^*$, then $\mathcal{V} \subset \mathcal{A}_\infty$. Since (E, A) is regular, $\mathcal{A}_\infty \cap \ker E = 0$. Thus we have $\mathcal{V} \cap \ker E = 0$. Together with the hypothesis that $\mathcal{V}$ is (E, A)-deflating, this implies that $\mathcal{V}$ is a solution space of (E, A). For any $x_0 \in \mathcal{V} \subset \mathcal{A}_i$, there exists $x_1 \in \mathcal{A}_{i-1}$ such that $Ex_1 = Ax_0$, thanks to the fact that $A\mathcal{A}_i \subset E\mathcal{A}_{i-1}$. By induction, we can find $x_k \in \mathcal{A}_{i-k}$, $k = 2, 3, \cdots, i$, such that $Ex_k = Ax_{k-1}$. In particular, $x_i = 0$ since $\mathcal{A}_0 = 0$. Let $x_k = 0$ for all $k \geq i + 1$. It is clear that $(x_k)_{k=1}^{\infty}$ is the fundamental sequence for x_0 which is terminating. $\square$

Corollary 3.15. *Suppose that (E, A) is regular. Then $\mathcal{A}_\infty$ is a terminating solution space of (E, A). In fact, $\mathcal{A}_\infty$ is the maximal terminating solution space, that is, if $\mathcal{V}$ is any terminating solution space of (E, A), then $\mathcal{V} \subset \mathcal{A}_\infty$.*

Proof. Since $A\mathcal{A}_\infty \subset E\mathcal{A}_\infty$, $\mathcal{A}_\infty$ is a solution space of (E, A). The assertion that $\mathcal{A}_\infty$ is terminating is an immediate application of Proposition 3.14. Let $\mathcal{V}$ be a terminating solution space of (E, A) and let $x_0 \in \mathcal{V}$. Let $(x_k)_{k=1}^{\infty}$ be the corresponding terminating fundamental sequence for x_0, with $x_k = 0$ for all $k \geq i$. It follows from $Ax_{i-1} = Ex_i = 0$ that $x_{i-1} \in \ker A = \mathcal{A}_1$. Similarly, by $Ax_{i-2} = Ex_{i-1} \in E\mathcal{A}_1$, we have $x_{i-2} \in \mathcal{A}_2$. Repeating the arguments we eventually come to the conclusion that $x_0 \in \mathcal{A}_i \subset \mathcal{A}_\infty$. This proves that $\mathcal{V} \subset \mathcal{A}_\infty$. $\square$

Chapter 4

Complete Sequences of (E, A)

In the previous chapter, we have shown that if (E, A) is regular and solvable for x_0, then the fundamental sequence for x_0 is unique. In particular, this implies that the unique fundamental sequence for $x_0 = 0$ is the zero sequence. However, for any state x_0, we may in general find more than one *finite* sequence $(x_i)_{i=1}^{l}$ satisfying $Ex_i = Ax_{i-1}$ for all $i \in J_l^*$. In this chapter, we shall investigate some important properties of these finite sequences, especially we shall find the maximal length among all these finite sequences.

We first give a formal definition.

Definition 4.1. *Let $x_0 = 0$. A finite sequence $(x_i)_{i=1}^{l}$ is called a complete sequence of (E, A) of length l if it satisfies $Ex_i = Ax_{i-1}$ for all $i \in J_l^*$, with $x_1 \neq 0$ and $x_l \notin A^{-1}\mathrm{im}E$.*

The requirement of $x_1 \neq 0$ in the definition is intended to ensure that there is no zeros in the initial terms of the sequence $(x_i)_{i=0}^{l}$ other than $x_0 = 0$. This prevents the length of a complete sequence from being mendacious.

Given $x \in \mathcal{X}$. $y \in \mathcal{X}$ is called an *immediate descendant* of x if $Ey = Ax$. Similarly, $z \in \mathcal{X}$ is called an *immediate ancestor* of x if $Ex = Az$. Hence a complete sequence $(x_i)_{i=1}^{l}$ of (E, A) is a sequence which satisfies the difference equation $Ex_i = Ax_{i-1}$ starting from zero but cannot be extended to a sequence with longer length, i.e., x_l has no immediate descendant.

Notice that there is no complete sequence of (E, A) if E is non-singular. Hence *we* *shall assume that E is singular, as well as (E, A) is regular, throughout this chapter.*

For later use we define $\mathcal{S}_i := \mathcal{E}_i \cap A^{-1}E\mathcal{E}_{i+1}$ for each $i \in \mathbb{N}$. It is clear that for all $i \in \mathbb{N}$ we have $\mathcal{S}_i \subset \mathcal{S}_{i+1}$ since $\mathcal{E}_i \subset \mathcal{E}_{i+1}$. Define $\mathcal{S}_\infty := \bigcup_{i=0}^{\infty} \mathcal{S}_i$ and let $\kappa := \text{ind}(E, A)$. Note that $\kappa \geq 1$ since E is assumed to be singular.

Lemma 4.2. *We have $\mathcal{S}_{\kappa+j} = \mathcal{S}_{\kappa-1}$ for all $j \in \mathbb{N}$. In particular, $\mathcal{S}_\infty = \mathcal{S}_\kappa = \mathcal{S}_{\kappa-1}$.*

Proof. $\mathcal{S}_\kappa = \mathcal{E}_\kappa \cap A^{-1}E\mathcal{E}_{\kappa+1} = \mathcal{E}_{\kappa+j} \cap A^{-1}E\mathcal{E}_{\kappa+j+1} = \mathcal{S}_{\kappa+j}$ for all $j \in \mathbb{N}^*$. Notice that $A^{-1}E\mathcal{E}_{\kappa+1} = A^{-1}E\mathcal{E}_\kappa \subset A^{-1}A\mathcal{E}_{\kappa-1} = \mathcal{E}_{\kappa-1} + \ker A$. Since (E, A) is regular, we find that $\mathcal{S}_\kappa = \mathcal{E}_\kappa \cap A^{-1}E\mathcal{E}_{\kappa+1} \subset \mathcal{E}_\kappa \cap (\mathcal{E}_{\kappa-1} + \ker A) = (\mathcal{E}_\kappa \cap \ker A) + \mathcal{E}_{\kappa-1} = \mathcal{E}_{\kappa-1}$. It follows that $\mathcal{S}_\kappa = \mathcal{E}_{\kappa-1} \cap \mathcal{S}_\kappa = \mathcal{E}_{\kappa-1} \cap \mathcal{E}_\kappa \cap A^{-1}E\mathcal{E}_{\kappa+1} = \mathcal{E}_{\kappa-1} \cap A^{-1}E\mathcal{E}_\kappa = \mathcal{S}_{\kappa-1}$. $\square$

Lemma 4.3. *Let $x_0 = 0$. Let $(x_i)_{i=1}^{l}$ be a sequence satisfying $Ex_i = Ax_{i-1}$ for all $i \in J_l^*$. Then $x_i \in \mathcal{E}_i$ for all $i \in J_l$. Moreover, if $x_1 \neq 0$, then for all $i \in J_l^*$, $x_i \in \mathcal{E}_i \setminus \mathcal{E}_{i-1}$, in particular $x_i \neq 0$.*

Proof. We prove the first statement by induction. Clearly $x_0 \in \mathcal{E}_0$. Suppose that $x_{i-1} \in \mathcal{E}_{i-1}$ for some $i \in J_l$. We have $Ex_i = Ax_{i-1} \in A\mathcal{E}_{i-1}$. This implies that $x_i \in E^{-1}A\mathcal{E}_{i-1} = \mathcal{E}_i$. Now, suppose that $x_1 \neq 0$ but $x_2 \in \mathcal{E}_1$. Then we have $0 = Ex_2 = Ax_1$. This means that $x_1 \in \ker A \cap \mathcal{E}_1 = 0$, which is a contradiction. Hence $x_2 \in \mathcal{E}_2 \setminus \mathcal{E}_1$. In particular, $x_2 \neq 0$. Inductively, assume that $x_{i-1} \notin \mathcal{E}_{i-2}$. Then $Ex_i = Ax_{i-1} \notin A\mathcal{E}_{i-2}$. This shows that $x_i \notin E^{-1}A\mathcal{E}_{i-2} = \mathcal{E}_{i-1}$. $\square$

Corollary 4.4. *Let $x_0 = 0$. Let $(x_i)_{i=1}^{l}$ be a complete sequence of (E, A) of length l. Then $x_i \in \mathcal{E}_i \setminus \mathcal{E}_{i-1}$ for all $i \in J_l^*$. In particular, $x_j \neq x_k$ for all $j \neq k$, $j, k \in J_l$.*

Proposition 4.5. *Let $x_0 = 0$. Suppose that $(x_i)_{i=1}^{l}$ is a complete sequence of (E, A) of length l. Then $x_i \in \mathcal{S}_i \setminus \mathcal{S}_{i-1}$ for all $i \in J_{l-1}^*$ but $x_l \notin \mathcal{S}_l$.*

Proof. Since we have proved that $x_i \in \mathcal{E}_i \setminus \mathcal{E}_{i-1}$ for all $i \in J_l^*$, we need only to prove that $x_i \in A^{-1}E\mathcal{E}_{i+1}$ for all $i \in J_{l-1}^*$ but $x_l \notin A^{-1}E\mathcal{E}_{l+1}$. By $Ax_i = Ex_{i+1} \in E\mathcal{E}_{i+1}$, we obtain $x_i \in A^{-1}E\mathcal{E}_{i+1}$. Finally, since $x_l \notin A^{-1}\text{im}E$, we find that $x_l \notin A^{-1}E\mathcal{E}_{l+1}$. This completes the proof. $\square$

Theorem 4.6. *The maximal length among all complete sequences of* (E, A) *equals the index of* (E, A).

Proof. Let l_m be the maximal length among all complete sequences of (E, A) and let $\kappa := \mathrm{ind}(E, A)$. Since $\kappa \geq 1$, $\mathcal{E}_\infty = \mathcal{E}_\kappa \neq 0$. Observe that for any $y \in \mathcal{E}_\kappa \setminus \mathcal{E}_{\kappa-1}$, there is a finite sequence $(x_i)_{i=0}^{\kappa}$ satisfying $Ex_i = Ax_{i-1}$ with $x_0 = 0$, $x_\kappa = y$, and $x_i \in \mathcal{E}_i \setminus \mathcal{E}_{i-1}$ for all $i \in J_\kappa^*$. This implies that $l_m \geq \kappa$. We shall prove that there does not exist any complete sequence of (E, A) of length larger than κ, and thus we can conclude that $l_m = \kappa$. To see this, we assume by contradiction that $(x_i)_{i=1}^{l}$ is a complete sequence of (E, A) of length $l > \kappa \geq 1$. Then, by Proposition 4.5, $x_{l-1} \in \mathcal{E}_{l-1} = \mathcal{E}_\kappa$. It follows from $Ex_{l-1} = Ax_{l-2}$ that $x_{l-2} \in A^{-1}E\mathcal{E}_\kappa \subset A^{-1}A\mathcal{E}_{\kappa-1} = \mathcal{E}_{\kappa-1} + \ker A$. Thus $x_{l-2} = e + a$ for some $e \in \mathcal{E}_{\kappa-1}$ and $a \in \ker A$. Also, by Proposition 4.5, $x_{l-2} \in \mathcal{E}_{l-2} \subset \mathcal{E}_{l-1} = \mathcal{E}_\kappa$. Consequently, $x_{l-2} - e = a \in \mathcal{E}_\kappa \cap \ker A = 0$. Hence we obtain that $x_{l-2} = e \in \mathcal{E}_{\kappa-1}$. By induction, it is easy to see that $x_{l-i} \in \mathcal{E}_{\kappa-i+1}$ for each $i \in J_{\kappa+1}^*$. In particular, $x_{l-\kappa-1} \in \mathcal{E}_0 = 0$, that is, $x_{l-\kappa-1} = 0$. By Corollary 4.4, this is a contradiction if $l > \kappa + 1$. On the other hand, if $l = \kappa + 1$, then we have $x_{l-1} \in \mathcal{S}_{l-1} = \mathcal{S}_\kappa = \mathcal{S}_{\kappa-1} \subset \mathcal{E}_{\kappa-1}$, provided by Lemma 4.2 and Proposition 4.5. It can be shown with similar arguments as above that $x_{l-i} \in \mathcal{E}_{\kappa-i}$. In particular, $x_{l-\kappa} = x_1 = 0$. Again, this is a contradiction. This completes the proof. $\square$

Example 4.7. *Let* $E = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$, $A = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$. *We want to find all complete sequences of* (E, A). *Let* e_i, $i \in J_5^*$, *be the column vector with 1 in the* i*-th component and 0 elsewhere. It is straightforward to compute that* $\mathcal{E}_1 = \ker E = \mathrm{span}\{e_3\} = A\mathcal{E}_1$, $\mathcal{E}_2 = \mathrm{span}\{e_3, e_4\} = A\mathcal{E}_2$, *and* $\mathcal{E}_3 = \mathcal{E}_4 = \mathcal{E}_\infty = \mathrm{span}\{e_3, e_4, e_5\}$. *Thus* $\mathrm{ind}(E, A) = 3$. *It is also straightforward to compute that* $\mathcal{S}_1 = \mathrm{span}\{e_3\}$, $\mathcal{S}_2 = \mathcal{S}_3 = \mathcal{S}_\infty = \mathrm{span}\{e_3, e_4\}$. *Now, let* $x_0 = 0$. *Solving* $Ex_1 = Ax_0$ *for* x_1 *gets* $x_1 = ae_3 \in \mathcal{E}_1 = \mathcal{S}_1$ *for any* $a \neq 0$. *Then solve* $Ex_2 = Ax_1$ *for* x_2 *to get* $x_2 = be_3 + ae_4 \in \mathcal{E}_2 = \mathcal{S}_2$ *for any* b. *We continue to solve* $Ex_3 = Ax_2$ *for* x_3 *to find that* $x_3 = ce_3 + be_4 + ae_5$. *Since* $a \neq 0$, *we* *find that $x_3 \notin \mathcal{S}_3$. Hence there does not exist x_4 such that $Ex_4 = Ax_3$. We thus conclude that the maximal length among all complete sequences is equal to* 3*, the index of* (E, A)*.*

Now we generalize the above results to any non-zero initial state x_0.

Definition 4.8. *Let $x_0 \neq 0$ and $x_0 \in A^{-1}\mathrm{im}E$. A finite sequence $(x_i)_{i=1}^{l}$ is called a* generalized complete sequence of (E, A) for x_0 of length l *if it satisfies $Ex_i = Ax_{i-1}$ for all $i \in J_l^*$, with $x_1 \notin \mathcal{E}_f$ and $x_l \notin A^{-1}\mathrm{im}E$. The maximal length among all generalized complete sequences of (E, A) for x_0 is denoted by $\rho(x_0)$. For convenience, we define $\rho(x_0) = 0$ if $x_0 \notin A^{-1}\mathrm{im}E$, i.e. x_0 has no immediate descendant.*

We first consider the case for $x_0 \in \mathcal{E}_f$.

Proposition 4.9. *Let $x_0 \in \mathcal{E}_f$ and let $(y_i)_{i=1}^{\infty}$ be the fundamental sequence for x_0. Then $(x_i)_{i=1}^{l}$ is a generalized complete sequence for x_0 of length l if and only if $x_i = y_i + z_i$ for all $i \in J_l^*$, where $(z_i)_{i=1}^{l}$ is a complete sequence of length l.*

Proof. Suppose that $(x_i)_{i=1}^{l}$ is a generalized complete sequence for x_0 of length l. For convenience, let $y_0 := x_0$ and $z_0 := 0$. It is easy to see that $E(x_i - y_i) = A(x_{i-1} - y_{i-1})$ for all $i \in J_l^*$. In particular. we have $E(x_1 - y_1) = 0$, i.e. $z_1 := x_1 - y_1 \in \mathcal{E}_1$. Similarly, we have $E(x_2 - y_2) = A(x_1 - y_1) = Az_1 \in A\mathcal{E}_1$, i.e. $z_2 := x_2 - y_2 \in E^{-1}A\mathcal{E}_1 = \mathcal{E}_2$. It can be shown by induction that for all $i \in J_l^*$ we have $z_i := x_i - y_i \in \mathcal{E}_i$ and $Ez_i = Az_{i-1}$. Note that $z_1 \neq 0$ since $x_1 \notin \mathcal{E}_f$ but $y_1 \in \mathcal{E}_f$. If $z_l \in A^{-1}\mathrm{im}E$, then we have $Az_l = Ax_l - Ay_l = Ew$ for some w, i.e. $Ax_l = Ay_l + Ew = E(y_{l+1} + w) \in \mathrm{im}E$. This implies that $x_l \in A^{-1}\mathrm{im}E$, which is a contradiction. Hence we have $z_l \notin A^{-1}\mathrm{im}E$. This shows that $(z_i)_{i=1}^{l}$ is a complete sequence of length l. Conversely, suppose that $x_i = y_i + z_i$ for all $i \in J_l^*$, where $(z_i)_{i=1}^{l}$ is a complete sequence of length l. Then for all $i \in J_l^*$ we have $Ex_i = Ey_i + Ez_i = Ay_{i-1} + Az_{i-1} = Ax_{i-1}$, where $y_0 := x_0$ and $z_0 := 0$. Also we have $x_1 \notin \mathcal{E}_f$ since $z_1 \notin \mathcal{E}_f$. Moreover, $x_l \notin A^{-1}\mathrm{im}E$ since $z_l \notin A^{-1}\mathrm{im}E$. This shows that $(x_i)_{i=1}^{l}$ is a generalized complete sequence for x_0 of length l. □

Combining Theorem 4.6 and Proposition 4.9, we immediately obtain the following result.

Corollary 4.10. $\rho(x_0) = \mathrm{ind}(E, A)$ *for all* $x_0 \in \mathcal{E}_f$.

Next, we consider the case for $x_0 \in \mathcal{E}_\infty$.

Proposition 4.11. *Let* $\kappa := \mathrm{ind}(E, A)$ *and let* $x_0 \in \mathcal{E}_j \setminus \mathcal{E}_{j-1}, 1 \le j \le \kappa$. *Then*

$$\rho(x_0) \le \kappa - j. \tag{4.1}$$

In particular, if $x_0 \in \mathcal{E}_\kappa \setminus \mathcal{E}_{\kappa-1}$, *then* $\rho(x_0) = 0$, *i.e.* $x_0 \notin A^{-1}\mathrm{im}E$.

Proof. If $x_0 \in \mathcal{E}_\kappa \setminus \mathcal{E}_{\kappa-1}$, then it follows immediately from Corollary 4.4 and Theorem 4.6 that $\rho(x_0) = 0$. Now, let $x_0 \in \mathcal{E}_j \setminus \mathcal{E}_{j-1}$ for some $j \in \mathbb{N}^*, 1 \le j < \kappa$. The case that $x_0 \notin A^{-1}\mathrm{im}E$ is trivial. So in what follows we let $x_0 \in A^{-1}\mathrm{im}E$. Let $(x_i)_{i=1}^l$ be any generalized complete sequence of (E, A) for x_0 of length $l \ge 1$. Since $x_0 \in \mathcal{E}_j \setminus \mathcal{E}_{j-1}$, there is a finite sequence $(y_i)_{i=0}^j$ satisfying $Ey_i = Ay_{i-1}$ with $y_0 = 0$, $y_j = x_0$, and $y_i \in \mathcal{E}_i \setminus \mathcal{E}_{i-1}$ for all $i \in J_j^*$. Let $y_{j+m} := x_m$ for all $m \in J_l^*$. It is easy to see that the finite sequence $(y_i)_{i=1}^{j+l}$ is a complete sequence of length $j + l$. By Theorem 4.6 we have $j + l \le \kappa$. This implies that $l \le \kappa - j$. □

Remark 4.12. *The bound provided by inequality (4.1) cannot, in general, be improved upon. For example, let* $\kappa > 1$ *and* $(x_i)_{i=1}^\kappa$ *be a complete sequence of maximal length* κ. *Then* $x_j \in \mathcal{E}_j \setminus \mathcal{E}_{j-1}$ *and* $(x_i)_{i=j+1}^\kappa$ *is a generalized complete sequence for* x_j *with* $\rho(x_j) = \kappa - j$.

The following theorem is the main result of this chapter. Recall that (E, A) is assumed to be regular, hence we have $\mathcal{X} = \mathcal{E}_f \oplus \mathcal{E}_\infty$.

Theorem 4.13. *Let* $x_0 \ne 0$ *and* $x_0 \in A^{-1}\mathrm{im}E$. *Let* x_0 *be decomposed as* $x_0 = f_0 + e_0$, *where* $f_0 \in \mathcal{E}_f$ *and* $e_0 \in \mathcal{E}_\infty$ *with* $e_0 \ne 0$. *Let* $\kappa = \mathrm{ind}(E, A)$. *Then* $\rho(x_0) = \rho(e_0)$.

Proof. Let $(x_i)_{i=1}^l$ be a generalized complete sequence for x_0 of possibly maximal length $l \ge 1$, and let $x_i = f_i + e_i$ with $f_i \in \mathcal{E}_f$, and $e_i \in \mathcal{E}_\infty$ for all $i \in J_l^*$. Now, since f_0 is solvable, there is a unique fundamental sequence $\{y_i\}_{i=1}^\infty$ for f_0, with $y_i \in \mathcal{E}_f$ for all $i \in \mathbb{N}^*$. In particular, we have that $E(f_1 + e_1 - y_1) = Ae_0$. This implies that

$f_1 + e_1 - y_1 \in \mathcal{E}_\infty$. So, $f_1 - y_1 \in \mathcal{E}_\infty \cap \mathcal{E}_f = 0$, i.e. $f_1 = y_1$. It can be shown by induction that $f_i = y_i$ for all $i \in J_l^*$. This in turn implies that $Ee_i = Ae_{i-1}$ for all $i \in J_l^*$. Moreover, $e_l \notin A^{-1}\mathrm{im}E$ since $x_l \notin A^{-1}\mathrm{im}E$. Thus $(e_i)_{i=1}^l$ is a generalized complete sequence of e_0. Therefore $\rho(x_0) \leq \rho(e_0)$.

On the other hand, since $0 \neq e_0 \in \mathcal{E}_\infty$, it can be shown with a similar argument as used in the proof of Proposition 4.11 that there exists a complete sequence $(h_i)_{i=1}^l$ of length $j + l$ for some $j \in \mathbb{N}^*$, $1 \leq j \leq \kappa$ with $e_0 = h_j$, $\rho(e_0) = l - j$. Hence we can find a partial complete sequence $(\tilde{x}_i)_{i=1}^{l-j}$ such that $\tilde{x}_i = y_i + h_{i+j}$ for $1 \leq i \leq l - j$. So the inequality $\rho(x_0) \geq \rho(e_0)$ is true. So we can conclude that $\rho(x_0) = \rho(e_0)$. $\square$

Theorem 4.14. *The maximal length among all generalized complete sequences of* (E, A) *for any* x_0 *is equal to or less than the index of* (E, A).

Now, let us consider some examples.

Example 4.15. *Let*

$$E = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

As in Example 4.7, we let e_i, $i \in J_5^*$, *be the column vector with 1 in the* i*-th component and 0 elsewhere. It is easy to see that* $\mathrm{ind}(E, A) = 2$. *Thus,* $\mathcal{E}_1 \subsetneq \mathcal{E}_2 = \mathcal{E}_\infty$. *It is straightforward to compute that* $\mathcal{E}_1 = \ker E = \mathrm{span}\{e_3, e_4\}$, *and* $A\mathcal{E}_1 = \mathrm{span}\{e_3, e_4\}$ *Now,*

$$\begin{pmatrix} x \\ y \\ z \\ w \\ u \end{pmatrix} \in \mathcal{E}_2 = E^{-1}A\mathcal{E}_1 \Leftrightarrow \begin{pmatrix} x \\ y \\ u \\ 0 \\ 0 \end{pmatrix} \in A\mathcal{E}_1 \Leftrightarrow x = y = 0 \textit{ So, } \mathcal{E}_2 = \mathrm{span}\{e_3, e_4, e_5\},$$

$A\mathcal{E}_2 = \text{span}\{e_3, e_4, e_5\}$*, and* $E\mathcal{E}_2 = \text{span}\{e_3\} \Rightarrow \begin{pmatrix} x \\ y \\ z \\ w \\ u \end{pmatrix} \in \mathcal{E}_3 = E^{-1}A\mathcal{E}_2 \Leftrightarrow \begin{pmatrix} x \\ y \\ u \\ 0 \\ 0 \end{pmatrix} \in$ $\text{span}\{e_3, e_4, e_5\} \Leftrightarrow x = y = 0 \Rightarrow E^{-1}A\mathcal{E}_2 = E^{-1}A\mathcal{E}_3 = \text{span}\{e_3, e_4, e_5\}$ *Moreover, if* $\begin{pmatrix} x \\ y \\ z \\ w \\ u \end{pmatrix} \in A^{-1}E\mathcal{E}_2 \Leftrightarrow \begin{pmatrix} 0 \\ y \\ z \\ w \\ u \end{pmatrix} \in E\mathcal{E}_2 \Leftrightarrow y = w = u = 0 \Rightarrow A^{-1}E\mathcal{E}_2 =$ $\text{span}\{e_1, e_3\} = A^{-1}E\mathcal{E}_3$ *Hence,* $\mathcal{S}_1 = \mathcal{E}_1 \cap A^{-1}E\mathcal{E}_2 = \text{span}\{e_3\}$*, and* $\mathcal{S}_2 = \mathcal{E}_2 \cap A^{-1}E\mathcal{E}_3 = \text{span}\{e_3\} = \mathcal{S}_3 = \cdots = \mathcal{S}_\infty$ *Let,* $x_0 = 0, x_1 = ae_3 + be_4 \in \mathcal{E}_1, a^2 + b^2 \neq 0$ *If* $b \neq 0 \Rightarrow \nexists x_2 \in Ex_2 = Ax_1$*. So,* $\exists x_2 \ni Ex_2 = Ax_1 \Leftrightarrow b = 0$ *and in the case* $x_2 = ae_5, a \neq 0$*. Note:* $x_2 \notin \mathcal{S}_2 \Rightarrow \nexists x_3 \ni Ex_3 = Ax_2 \Rightarrow \{x_1 = ae_3, x_2 = ae_5 + be_3\}$ *is a complete sequence of* (E, A) *of maximal length* $2 = \text{ind}(E, A)$*, and* $\{x_1 = be_4\}$ *is a complete sequence of* (E, A) *of length* 1 *with* $b \neq 0$*.*

Example 4.16. *We can do more for above example. Consider* $x_0 = e_1 + e_3$*, then all possible* x_1 *have the type* $u + e_5$ *for* $u \in \ker E$*. Then* $A(u + e_5) = u + e_5 \notin \text{im}E$*, as the same for which we said before.*

Chapter 5

Solvability of (E, A, B)

Definition 5.1. *Let an initial state x_0 and a control input (control sequence) $u = (u_k)_{k=0}^{\infty}$ be given. (E, A, B) is said to be* solvable *for (x_0, u) if there exists a sequence $(x_k)_{k=1}^{\infty}$ satisfying $Ex_{k+1} = Ax_k + Bu_k$ for all $k \in \mathbb{N}$; in this case, the sequence $(x_k)_{k=1}^{\infty}$ is called a* solution sequence *of (E, A) for (x_0, u).*

Proposition 5.2. *Suppose that (E, A) is regular. If (E, A) is solvable for (x_0, u), then the solution sequence for (x_0, u) is uniquely determined by*

$$x_k = V_f \Lambda^k V_f^+ x_0 + \sum_{i=0}^{k-1} V_f \Lambda^{k-i-1} B_1 u_i - \sum_{i=0}^{p-1} V_\infty N^i B_2 u_{k+i} \tag{5.1}$$

for all $k \in \mathbb{N}^*$, where V_f and V_∞ are basis matrices for $\mathcal{E}_f$ and $\mathcal{E}_\infty$, respectively, Λ and N are matrices satisfying $EV_f\Lambda = AV_f$ and $EV_\infty = AV_\infty N$, N is nilpotent of index p, V_f^+ denotes the Moore-Penrose inverse of V_f, and $\begin{pmatrix} B_1 \\ B_2 \end{pmatrix} := \begin{pmatrix} EV_f & AV_\infty \end{pmatrix}^{-1} B.$

Proof. Let $(x_k)_{k=1}^{\infty}$ and $(y_k)_{k=1}^{\infty}$ be solution sequences for (x_0, u). Set $y_0 := x_0$. Then we have $E(y_{k+1} - x_{k+1}) = A(y_k - x_k)$ for all $k \in \mathbb{N}$. By Corollary 3.8, $y_k = x_k$ for all $k \in \mathbb{N}^*$. This proves the uniqueness of solution sequence. It is straightforward to verify that $(x_k)_{k=1}^{\infty}$ given in (5.1) satisfies $Ex_{k+1} = Ax_k + Bu_k$. $\square$

Definition 5.3. *We say that a subspace $\mathcal{V}$ of $\mathcal{X}$ is a* solution space of (E, A) modulo im$B$

if for every $x_0 \in \mathcal{V}$, there exists a control sequence $u = (u_k)_{k=0}^{\infty}$ such that (E, A, B) is solvable for (x_0, u) with a solution sequence $(x_k)_{k=1}^{\infty}$ in $\mathcal{V}$.

A solution space of (E, A) modulo $\mathrm{im}B$ will also be called an (E, A, B) *controlled solution space* or a *controlled solution space of* (E, A, B), or more simply a *controlled solution space* when the triple (E, A, B) is understood. Clearly, a solution space of (E, A) is a solution space of (E, A) modulo $\mathrm{im}B$. Also, the sum of finitely many solution spaces of (E, A) modulo $\mathrm{im}B$ is still a solution space of (E, A) modulo $\mathrm{im}B$.

Lemma 5.4. *$\mathcal{V}$ is a solution space of (E, A) modulo $\mathrm{im}B$ if and only if $A\mathcal{V} \subset E\mathcal{V} + \mathrm{im}B$.*

Proof. Suppose that $\mathcal{V}$ is a solution space of (E, A) modulo $\mathrm{im}B$. Then for any $x_0 \in \mathcal{V}$, there exists $x_1 \in \mathcal{V}$ and $u_0 \in \mathcal{U}$ such that $Ax_0 = Ex_1 - Bu_0 \in E\mathcal{V} + \mathrm{im}B$. This implies that $A\mathcal{V} \subset E\mathcal{V} + \mathrm{im}B$.
On the other hand, suppose that $A\mathcal{V} \subset E\mathcal{V} + \mathrm{im}B$. Let $x_0 \in \mathcal{V}$. Then there exist $x_1 \in \mathcal{V}$ and $u_0 \in \mathcal{U}$ such that $Ax_0 = Ex_1 - Bu_0$. By induction, we can prove that for $x_k \in \mathcal{V}$, there exist $x_{k+1} \in \mathcal{V}$ and $u_k \in \mathcal{U}$ such that $Ax_k = Ex_{k+1} - Bu_k$ for all $k \in \mathbb{N}$. $\square$

Theorem 5.5. *Let $\mathcal{K}$ be a subspace of $\mathcal{X}$. Then there exists a unique maximal solution space of (E, A) modulo $\mathrm{im}B$ contained in $\mathcal{K}$.*

Proof. Let $\mathcal{S}$ be the set of all solution spaces of (E, A) modulo $\mathrm{im}B$ contained in $\mathcal{K}$. $\mathcal{S}$ is non-empty since it contains the zero subspace. Partially order $\mathcal{S}$ by set theoretic inclusion. Let $\mathcal{C} = \{\mathcal{V}_i | i \in I\}$ be any chain in $\mathcal{S}$ of solution spaces of (E, A) modulo $\mathrm{im}B$ contained in $\mathcal{K}$ and let $\mathcal{V}' := \bigcup_{i \in I} \mathcal{V}_i$. For any $x_0 \in \mathcal{V}'$, $x_0 \in \mathcal{V}_i$ for some $i \in I$. Since $\mathcal{V}_i \in \mathcal{S}$, there exist a control sequence $(u_k)_{k=0}^{\infty}$, and a sequence $(x_k)_{k=1}^{\infty}$, with $x_k \in \mathcal{V}_i \subset \mathcal{V}'$, satisfying $Ex_{k+1} = Ax_k + Bu_k$ for all $k \in \mathbb{N}$. This shows that $\mathcal{V}' \in \mathcal{S}$. Clearly $\mathcal{V}'$ is an upper bound of the chain $\mathcal{C}$. By Zorn's lemma, $\mathcal{S}$ contains a maximal element $\mathcal{V}^*$. We claim that $\mathcal{V} \subset \mathcal{V}^*$ for every $\mathcal{V} \in \mathcal{S}$, thus showing that $\mathcal{V}^*$ is the unique maximal element in $\mathcal{S}$. Suppose, by contradiction, that there exists $\mathcal{V} \in \mathcal{S}$ not contained in $\mathcal{V}^*$. Then $\mathcal{V}^* \subsetneqq \mathcal{V} + \mathcal{V}^* \in \mathcal{S}$, contradicting that $\mathcal{V}^*$ is a maximal element in $\mathcal{S}$. $\square$

The maximal solution space of (E, A) modulo $\mathrm{im}B$ contained in a subspace $\mathcal{K}$ will be denoted by $\mathcal{K}_\infty(E, A, B)$ or simply $\mathcal{K}_\infty$ if E, A, B are understood. The following theorem suggests a recursive algorithm to obtain $\mathcal{K}_\infty$.

Theorem 5.6. *Let $\mathcal{K}$ be a subspace of $\mathcal{X}$. Define a sequence of subspaces $\mathcal{K}_i$ recursively by $\mathcal{K}_0 := \mathcal{K}$, and $\mathcal{K}_i := \mathcal{K} \cap A^{-1}(E\mathcal{K}_{i-1} + \mathrm{im}B)$ for all $i \in \mathbb{N}^*$. Then for each $i \in \mathbb{N}$ we have $\mathcal{K}_i \supset \mathcal{K}_{i+1}$, and there exists a smallest integer r, with $r \leq \mathrm{dim}\mathcal{K}$, such that $\mathcal{K}_{r+j} = \mathcal{K}_r$ for all $j \in \mathbb{N}^*$. Moreover, $\mathcal{K}_\infty = \bigcap_{i=0}^{\infty} \mathcal{K}_i = \mathcal{K}_r$.*

Proof. We prove $\mathcal{K}_i \supset \mathcal{K}_{i+1}$ by induction. It is clear that $\mathcal{K}_0 \supset \mathcal{K}_1$. Suppose that $\mathcal{K}_{i-1} \supset \mathcal{K}_i$ for some $i \in \mathbb{N}^*$. Then

$$\mathcal{K}_{i+1} = \mathcal{K} \cap A^{-1}(E\mathcal{K}_i + \mathrm{im}B) \subset \mathcal{K} \cap A^{-1}(E\mathcal{K}_{i-1} + \mathrm{im}B) = \mathcal{K}_i.$$

By the finiteness of dimension of $\mathcal{X}$, there exists a smallest integer r such that $\mathcal{K}_r = \mathcal{K}_{r+1}$. If $\mathcal{K}_{r+j} = \mathcal{K}_r$ for some $j \in \mathbb{N}^*$, we have

$$\mathcal{K}_{r+j+1} = \mathcal{K} \cap A^{-1}(E\mathcal{K}_{r+j} + \mathrm{im}B) = \mathcal{K} \cap A^{-1}(E\mathcal{K}_r + \mathrm{im}B) = \mathcal{K}_{r+1} = \mathcal{K}_r.$$

By induction, this proves that $\mathcal{K}_{r+j} = \mathcal{K}_r$ for all $j \in \mathbb{N}^*$. Clearly, $\mathcal{K}_r = \bigcap_{i=0}^{\infty} \mathcal{K}_i$. In addition, $\mathcal{K}_r$ is a solution space of (E, A) modulo $\mathrm{im}B$ contained in $\mathcal{K}$ because $\mathcal{K}_r \subset \mathcal{K}$ and

$$A\mathcal{K}_r = A\mathcal{K}_{r+1} = A(\mathcal{K} \cap A^{-1}(E\mathcal{K}_r + \mathrm{im}B)) \subset AA^{-1}(E\mathcal{K}_r + \mathrm{im}B) \subset E\mathcal{K}_r + \mathrm{im}B.$$

Moreover, let $\mathcal{V}$ be any solution space of (E, A) modulo $\mathrm{im}B$ contained in $\mathcal{K}$. We have $\mathcal{V} \subset \mathcal{K} = \mathcal{K}_0$. Assume that $\mathcal{V} \subset \mathcal{K}_{i-1}$ for some $i \in \mathbb{N}^*$. Since $A\mathcal{V} \subset E\mathcal{V} + \mathrm{im}B$, we have $\mathcal{V} \subset A^{-1}(E\mathcal{V} + \mathrm{im}B)$. Therefore $\mathcal{V} \subset \mathcal{K} \cap A^{-1}(E\mathcal{V} + \mathrm{im}B) \subset \mathcal{K} \cap A^{-1}(E\mathcal{K}_{i-1} + \mathrm{im}B) = \mathcal{K}_i$. By induction again, we thus prove that $\mathcal{V} \subset \mathcal{K}_i$ for all $i \in \mathbb{N}$. In particular, $\mathcal{V} \subset \mathcal{K}_r$. Hence $\mathcal{K}_\infty = \mathcal{K}_r$. Finally, since $\mathrm{dim}\mathcal{K} > \mathrm{dim}K_1 > \cdots > \mathrm{dim}\mathcal{K}_r \geq 0$, we have $r \leq \mathrm{dim}\mathcal{K}$. $\square$

Now, we can use Theorem 5.6 to compute $\mathcal{X}_\infty$, the maximal solution space of (E, A) modulo $\mathrm{im}B$

Corollary 5.7. *Define $\mathcal{X}_0 := \mathcal{X}$, and $\mathcal{X}_i := A^{-1}(E\mathcal{X}_{i-1} + \mathrm{im}B)$ for all $i \in \mathbb{N}^*$. Then for each $i \in \mathbb{N}$ we have $\mathcal{X}_i \supset \mathcal{X}_{i+1}$, and there exists a smallest integer r, with $r \leq n$, such that $\mathcal{X}_{r+j} = \mathcal{X}_r$ for all $j \in \mathbb{N}^*$. Moreover, $\mathcal{X}_\infty = \bigcap_{i=0}^{\infty} \mathcal{X}_i = \mathcal{X}_r = A^{-1}(E\mathcal{X}_\infty + \mathrm{im}B)$.*

Proof. Let $\mathcal{K} = \mathcal{X}$ in Theorem 5.6. The last equation hold since $\mathcal{X}_\infty = \bigcap_{i=0}^{\infty} \mathcal{X}_i = \mathcal{X}_r = \mathcal{X}_{r+1} = A^{-1}(E\mathcal{X}_r + \mathrm{im}B) = A^{-1}(E\mathcal{X}_\infty + \mathrm{im}B)$. □

Remark 5.8. *If we define $\mathcal{C}_0 := A^{-1}\mathrm{im}B$ and $\mathcal{C}_i := A^{-1}E\mathcal{C}_{i-1} + A^{-1}\mathrm{im}B$ for all $i \in \mathbb{N}^*$, then we have $\mathcal{C}_i \subset \mathcal{C}_{i+1}$ for all $i \in \mathbb{N}$. Since $\mathcal{X}$ has finite dimension, $\mathcal{C}_m = \mathcal{C}_{m+1}$ for some $m \in \mathbb{N}$. It can be easily shown that $\mathcal{C}_{m+j} = \mathcal{C}_m$ for all $j \in \mathbb{N}^*$. Set $\mathcal{C}_\infty := \bigcup_{i=0}^{\infty} \mathcal{C}_i$. Then $\mathcal{C}_\infty = \mathcal{C}_m = A^{-1}E\mathcal{C}_\infty + A^{-1}\mathrm{im}B$ and $\mathcal{C}_\infty$ is a solution space of (E, A) modulo $\mathrm{im}B$ since $A\mathcal{C}_\infty = A(A^{-1}E\mathcal{C}_\infty + A^{-1}\mathrm{im}B) = AA^{-1}E\mathcal{C}_\infty + AA^{-1}\mathrm{im}B \subset E\mathcal{C}_\infty + \mathrm{im}B$. Hence $\mathcal{E}_f + \mathcal{C}_\infty$ is also a solution space of (E, A) modulo $\mathrm{im}B$. As a result, we have $\mathcal{E}_f + \mathcal{C}_\infty \subset \mathcal{X}_\infty$.*

Corollary 5.9. *If $\det A \neq 0$, then the maximal solution space of (E, A) modulo $\mathrm{im}B$ is $\mathcal{E}_f + \mathcal{C}_\infty$.*

Proof. Since (E, A) is regular, $\mathcal{X} = \mathcal{E}_\infty + \mathcal{E}_f$. Note that $\mathcal{E}_\infty = \mathcal{E}_l$ for some l. With $E\mathcal{E}_i \subset A\mathcal{E}_{i-1}$, we have $A^{-1}E\mathcal{E}_i \subset A^{-1}A\mathcal{E}_{i-1} = \mathcal{E}_{i-1} + \ker A$. Since $\mathcal{X}_1 = (A^{-1})(E(\mathcal{E}_\infty + \mathcal{E}_f) + imB) \subset (A^{-1})((E\mathcal{E}_\infty + E\mathcal{E}_f) + imB) \subset A^{-1}E\mathcal{E}_\infty + (A^{-1})E\mathcal{E}_f + (A^{-1})imB =: \mathcal{M}_1$. If we define $\mathcal{M}_r := (A^{-1}E)^r\mathcal{E}_\infty + (A^{-1}E)^r\mathcal{E}_f + \mathcal{C}_r$, we want to claim that $\mathcal{X}_r \subset \mathcal{M}_r$ by induction. Thus, we have shown that it's true for $r = 1$. Assume it true for $r = n - 1$. Now when $r = n$, $\mathcal{X}_n = (A^{-1})(E(\mathcal{X}_{n-1}) + imB) \subset (A^{-1})(E(\mathcal{M}_{n-1}) + imB) \subset (A^{-1})(E((A^{-1}E)^{n-1}\mathcal{E}_\infty + (A^{-1}E)^{n-1}\mathcal{E}_f + \mathcal{C}_{n-1}) + imB) \subset (A^{-1}E)^n\mathcal{E}_\infty + (A^{-1}E)^n\mathcal{E}_f + \mathcal{C}_n$. So, we have shown the claim. By Remark 3.10, $\mathcal{X}_r \subset \mathcal{M}_r = (A^{-1}E)^r\mathcal{E}_\infty + (A^{-1}E)^r\mathcal{E}_f + \mathcal{C}_r \subset \mathcal{E}_{l-r} + \mathcal{E}_f + \mathcal{C}_r$. And for $r = l$, $A^{-1}E\mathcal{E}_1 = A^{-1}E\ker E = \ker A \subset \mathcal{A}_\infty \subset \mathcal{E}_f$ by Theorem 3.9 and Corollary3.15. Hence, $\mathcal{X}_l \subset \ker A + \mathcal{E}_f + \mathcal{C}_l = \mathcal{E}_f + \mathcal{C}_l$. So, as r approaches to infinity, we can get $\mathcal{X}_\infty \subset \mathcal{E}_f + \mathcal{C}_\infty$. But notice that we have shown that $\mathcal{E}_f + \mathcal{C}_\infty \subset \mathcal{X}_\infty$. Hence $\mathcal{X}_\infty = \mathcal{E}_f + \mathcal{C}_\infty$ □

Remark 5.10. *If $\det A \neq 0$ and $B = O$. Then the maximal solution subspace of (E, A) module $\mathrm{im}B$ is the same as the maximal solution subspace of (E, A). But notice that* *when $B = O$, $\mathcal{C}_\infty = \mathcal{A}_\infty \subset \mathcal{E}_f$. Hence $\mathcal{X}_\infty = \mathcal{E}_f$. So the conclusion is the same as before.*

Example 5.11. *If we consider A for $\det A \neq 0$, we may assume $A = I$. Let*

$$E = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

It can be easy to check that $\mathcal{X}_1 = \operatorname{span}\{e_1, e_2, e_3, e_4\}$, $\mathcal{X}_2 = \operatorname{span}\{e_1, e_2, e_3 + e_4\} = \mathcal{X}_\infty$. On the other hand, $\mathcal{C}_0 = \operatorname{span}\{e_1\} = \mathcal{C}_\infty$. $\mathcal{E}_f = \operatorname{span}\{e_1, e_2, e_3 + e_4\}$. Hence $\mathcal{X}_\infty = \mathcal{E}_f + \mathcal{C}_\infty$.

Now, we want to use the Drazin inverse to describe the maximal solvability space more clearly.

Lemma 5.12. *If (E, A) is regular, and $\tilde{E}_c := (cE - A)^{-1}E$, $\tilde{A}_c := (cE - A)^{-1}A$, $f_c(k) := (cE - A)^{-1}Bu_k$ for $c \notin \sigma_f(E, A)$. Then*

$$\begin{aligned} \tilde{E}_\mu{}^D \tilde{E}_\mu &= \tilde{E}_\lambda{}^D \tilde{E}_\lambda \\ \tilde{E}_\mu{}^D \tilde{A}_\mu &= \tilde{E}_\lambda{}^D \tilde{A}_\lambda \\ \tilde{A}_\mu{}^D \tilde{E}_\mu &= \tilde{A}_\lambda{}^D \tilde{E}_\lambda \\ \tilde{A}_\mu{}^D f_\mu(k) &= \tilde{A}_\lambda{}^D f_\lambda(k) \\ \tilde{E}_\mu{}^D f_\mu(k) &= \tilde{E}_\lambda{}^D f_\lambda(k) \end{aligned} \tag{5.2}$$

for $\mu, \lambda \notin \sigma_f(E, A)$.

Proof. See [12], Theorem 9.2.2. $\square$

Remark 5.13. *Assume (E, A) is regular. And let $\tilde{E}_c := (cE - A)^{-1}E$, $\tilde{A}_c := (cE - A)^{-1}A$, $\tilde{B}_c := (cE - A)^{-1}B$ for $c \notin \sigma_f(E, A)$. Then $\tilde{E}_c$ and $\tilde{A}_c$ commute, and $(\tilde{E}_c, \tilde{A}_c)$ is also regular. Let $f_c(k) := (cE - A)^{-1}Bu_k$, we may replace the system $Ex_{k+1} = Ax_k + Bu_k$ by $\tilde{E}_c x_{k+1} = \tilde{A}_c x_k + f_c(k)$.*

From now, we may drop the subscript c for the terms $\tilde{E_c}^D\tilde{E}_c$, $\tilde{E_c}^D\tilde{A}_c$, $\tilde{A_c}^D f_c(k)$ (or $\tilde{A_c}^D\tilde{B}_c)$) and $\tilde{E_c}^D f_c(k)$ (or $\tilde{E_c}^D\tilde{B}_c$).

Proposition 5.14. *If (E, A) is regular, and the system $Ex_{k+1} = Ax_k + Bu_k$ is solvable for x_0 with some $(u_k)_{k\in\mathbb{N}}$, then the solution sequence x_0 is uniquely determined by*

$$\begin{aligned} x_k = (\tilde{E}^D\tilde{A})^k\tilde{E}^D\tilde{E}x_0 + \sum_{i=0}^{k-1}(\tilde{E}^D\tilde{A})^{k-i-1}\tilde{E}^D f(i) \\ -(I - \tilde{E}^D\tilde{E})\sum_{i=0}^{p-1}(\tilde{E}\tilde{A}^D)^i\tilde{A}^D f(k+i) \end{aligned} \tag{5.3}$$

for all $k \in \mathbb{N}^*$ with $p = ind(E, I)$.

In particular, The initial state x_0 is solvable if and only if $x_0 \in \mathrm{im}\tilde{E}^D + \mathrm{im}(I - \tilde{E}^D\tilde{E})[I(\tilde{E}\tilde{A}^D)\cdots(\tilde{E}\tilde{A}^D)^{(p-1)}]\tilde{A}^D\tilde{B}$. More generally, x_0 is solvable if and only if $x_0 \in \mathrm{im}\tilde{E}^D + \mathrm{im}[I(\tilde{E}\tilde{A}^D)\cdots(\tilde{E}\tilde{A}^D)^{(p-1)}]\tilde{A}^D\tilde{B}$. Notice that $\mathrm{im}\tilde{E}^D = \mathrm{im}\tilde{E}^D\tilde{E}$.

Proof. See [12], theorem 9.3.2. □

Proposition 5.15. *Let $\mathcal{D}_0 := \mathcal{E}_f$, $\mathcal{D}_{i+1} := A^{-1}\{E\mathcal{D}_i + \mathrm{im}B\}$. Then $\mathcal{D}_{i+1} \supset \mathcal{D}_i$, for $i \leqslant \mathrm{ind}(E)$. And the maximal solvability space is $\mathcal{X}_\infty = \mathcal{D}_p$ where $p = \mathrm{ind}(E, I)$.*

Proof. Let $\lambda \neq 0$, and $\lambda \notin \sigma_f(E, A)$. By equation 5.3, we have the fact that x_0 is solvable for the system $\tilde{E}_\lambda x_{k+1} = \tilde{A}_\lambda x_k + \tilde{B}_\lambda u_k$ if and only if

$$x_0 = \tilde{E}_\lambda^D\tilde{E}_\lambda y - \sum_{i=0}^{p-1}(\tilde{E}_\lambda\tilde{A}_\lambda^D)^i\tilde{A}_\lambda^D\tilde{B}_\lambda u_i \tag{5.4}$$

for some $\{u_i\}_{i=1\ldots p-1}$ and for some y.

Then, since $\tilde{A}_\lambda^D \tilde{A}_\lambda (I - \tilde{E}_\lambda^D \tilde{E}_\lambda) = (I - \tilde{E}_\lambda^D \tilde{E}_\lambda)$, we have

$$\begin{aligned} \tilde{A}_\lambda (I - \tilde{E}_\lambda^D \tilde{E}_\lambda) x_0 &= (I - \tilde{E}_\lambda^D \tilde{E}_\lambda) \sum_{i=0}^{p-1} (\tilde{E}_\lambda \tilde{A}_\lambda^D)^i \tilde{B}_\lambda u_k \\ \Rightarrow \tilde{A}_\lambda x_0 - \sum_{i=0}^{p-1} (\tilde{E}_\lambda \tilde{A}_\lambda^D)^i \tilde{B}_\lambda u_k &\in \ker(I - \tilde{E}_\lambda^D \tilde{E}_\lambda) = R(\tilde{E}_\lambda^D) \\ \Rightarrow \tilde{A}_\lambda x_0 &\in \mathrm{im} \tilde{E}_\lambda^D + \mathrm{im}[I \ (\tilde{E}_\lambda \tilde{A}_\lambda^D) \cdots (\tilde{E}_\lambda \tilde{A}_\lambda^D)^{(p-1)}] \tilde{B}_\lambda \end{aligned}$$

Notice that $\mathrm{im} \tilde{E}_\lambda^D = \mathcal{E}_f$.(This is independent of choice of λ by using lemma 5.12) And we have known that $A\mathcal{E}_f \subset E\mathcal{E}_f$.

Hence

$$\begin{aligned} Ax_0 &\in (\lambda E - A)\mathcal{E}_f + \mathrm{im}B + \mathrm{im}[E \ E(\tilde{A}_\lambda^D \tilde{E}_\lambda) \cdots E(\tilde{A}_\lambda^D \tilde{E}_\lambda)^{(p-2)}] \tilde{A}_\lambda^D \tilde{B}_\lambda \\ \Rightarrow Ax_0 &\in E\{\mathcal{E}_f + \mathrm{im}[I \ (\tilde{A}_\lambda^D \tilde{E}_\lambda) \cdots (\tilde{A}_\lambda^D \tilde{E}_\lambda)^{(p-2)}] \tilde{A}_\lambda^D \tilde{B}_\lambda\} + \mathrm{im}B \\ \Rightarrow x_0 &\in A^{-1}\{E\{\mathcal{E}_f + \mathrm{im}[I \ (\tilde{A}_\lambda^D \tilde{E}_\lambda) \cdots (\tilde{A}_\lambda^D \tilde{E}_\lambda)^{(p-2)}] \tilde{A}_\lambda^D \tilde{B}_\lambda\} + \mathrm{im}B\} \end{aligned}$$

Now, let $\mathcal{U}_i := \mathcal{E}_f + \mathrm{im}[I \ (\tilde{A}_\lambda^D \tilde{E}_\lambda) \cdots (\tilde{A}_\lambda^D \tilde{E}_\lambda)^{(p-i-1)}] \tilde{A}_\lambda^D \tilde{B}_\lambda$. Similarly as before, we have that

$$\mathcal{U}_i = A^{-1}\{E\mathcal{U}_{i+1} + \mathrm{im}B\} \quad i \in J_{p-1} \tag{5.5}$$

In particular, $\mathcal{U}_p = \mathcal{E}_f$. Then we can find that $\mathcal{U}_i = \mathcal{D}_{p-i}$ for $i = 0, \ldots, p$. Hence we can conclude that $\mathcal{D}_{i+1} \supset \mathcal{D}_i$, and $\mathcal{X}_\infty \subseteqq \mathcal{D}_p$ where $p = \mathrm{ind}(E)$.

Conversely, if $x_0 \in \mathcal{D}_p$, then x_0 is solvable for $Ex_{k+1} = Ax_k + Bu_k$ by definition of $\mathcal{D}_p$ and $\mathcal{E}_f$ is a solution space of (E, A). Thus, $x_0 \in \mathcal{X}_\infty$.

□

Remark 5.16. *We can also find that $\mathcal{D}_l = \mathcal{D}_p$ for all $l \geqslant p$ where $p = \mathrm{ind}(E)$. Hence we can set $\mathcal{D}_\infty := \bigcup_{i=0}^\infty \mathcal{D}_i$. So, $\mathcal{X}_\infty = \mathcal{D}_\infty$ by Proposition 5.15.*

Example 5.17. *Let* $E = \begin{pmatrix} 3 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -3 & 1 \\ 0 & 2 & 3 & 0 & 2 \\ 0 & -1 & 0 & 3 & -1 \\ 0 & 2 & 0 & 6 & -1 \end{pmatrix}$, $A = \begin{pmatrix} 0 & 1 & 11/2 & 5 & 1 \\ 0 & 4 & 4 & 5 & 1 \\ 0 & 2 & 8 & 4 & 2 \\ 0 & -1 & -4 & -2 & -1 \\ 0 & -1 & -10 & -5 & -1 \end{pmatrix}$.

Then, $\mathrm{ind}(E, A) = 2 \Rightarrow \mathcal{E}_1 \subsetneqq \mathcal{E}_2 = \mathcal{E}_\infty$ *We can easy to check that* $\mathcal{E}_1 = \ker E = \mathrm{span}\{\begin{pmatrix} 1 & 1 & 2 & -1 & -4 \end{pmatrix}^T\}$ *and* $\mathcal{E}_2 = \mathrm{span}\{\begin{pmatrix} 1 & 1 & 2 & -1 & -4 \end{pmatrix}^T, \begin{pmatrix} 0 & -1 & 0 & 0 & 2 \end{pmatrix}^T\} = \mathcal{E}_\infty$*, and* $\mathcal{E}_f = \mathrm{span}\{e_1, \begin{pmatrix} -2 & -2 & -1 & 2 & 2 \end{pmatrix}^T, e_5\}$ *where* v^T *is denoted the transport vector of* v*. Now, assume that*

$$B = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

So, $\mathrm{im}B = \mathrm{span}\{e_1\}$*. Hence* $\mathcal{C}_0 = A^{-1}\mathrm{im}B = \mathrm{span}\{e_1, \begin{pmatrix} -2 & -2 & -1 & 2 & 2 \end{pmatrix}^T\}$*,*
$\mathcal{C}_1 = A^{-1}E\mathcal{C}_0 + A^{-1}\mathrm{im}B = \mathrm{span}\{e_1, \begin{pmatrix} -2 & -2 & -1 & 2 & 2 \end{pmatrix}^T\} = \mathcal{C}_0$*. i.e.* $\mathcal{C}_0 = \mathcal{C}_\infty$
Now, if we assume that $\mathcal{X}_0 = \mathcal{X}$*,* $\mathcal{X}_1 := A^{-1}(E\mathcal{X} + \mathrm{im}B) = \mathrm{span}\{e_1, \begin{pmatrix} -2 & -2 & -1 & 2 & 2 \end{pmatrix}^T, \begin{pmatrix} 1 & 1 & 2 & -1 & -4 \end{pmatrix}^T, e_5\}$*,* $\mathcal{X}_2 := A^{-1}(E\mathcal{X}_1 + \mathrm{im}B) = \mathrm{span}\{e_1, \begin{pmatrix} -2 & -2 & -1 & 2 & 2 \end{pmatrix}^T, e_5\}$*, and* $\mathcal{X}_3 = \mathcal{X}_2$*. Hence,* $\mathcal{X}_\infty = \mathrm{span}\{e_1, \begin{pmatrix} -2 & -2 & -1 & 2 & 2 \end{pmatrix}^T, e_5\} = \mathcal{E}_f$*.*

On the other hand, if $\lambda = 2$ *we can easy to check that* $\lambda \neq 0$*, and* $\lambda \notin \sigma_f(E, A)$*. We can find that*

$$\tilde{E}_\lambda = \begin{pmatrix} 1/2 & 1/3 & -5/24 & 5/12 & 0 \\ 0 & 1/3 & -1/3 & -1/3 & 0 \\ 0 & 2/3 & -1/6 & 1/3 & 0 \\ 0 & -1/3 & 1/3 & 1/3 & 0 \\ 0 & 2/3 & 7/3 & 4/3 & 1 \end{pmatrix}, \quad \tilde{E}_\lambda^D = \begin{pmatrix} 1/2 & 0 & -5/24 & 1/12 & 0 \\ 0 & 0 & -1/3 & -2/3 & 0 \\ 0 & 0 & -1/6 & -1/3 & 0 \\ 0 & 0 & 1/3 & 2/3 & 0 \\ 0 & 2 & 7/3 & 8/3 & 1 \end{pmatrix},$$

$$\tilde{A}_\lambda = \begin{pmatrix} 0 & 2/3 & -5/12 & 5/6 & 0 \\ 0 & -1/3 & -2/3 & -2/3 & 0 \\ 0 & 4/3 & -4/3 & 2/3 & 0 \\ 0 & -2/3 & 4/3 & -1/3 & 0 \\ 0 & 4/3 & 14/3 & 8/3 & 1 \end{pmatrix}, \quad \tilde{A}_\lambda^D = \begin{pmatrix} 0 & 0 & -5/12 & 1/6 & 0 \\ 0 & 1 & -2/3 & -4/3 & 0 \\ 0 & 0 & -4/3 & -2/3 & 0 \\ 0 & 0 & 2/3 & 1/3 & 0 \\ 0 & 4 & 14/3 & 5 & 1 \end{pmatrix}.$$

So, we can check that $\mathrm{im}\tilde{E}_\lambda^D + \mathrm{im}[I(\tilde{E}_\lambda\tilde{A}_\lambda^D)\cdots(\tilde{E}_\lambda\tilde{A}_\lambda^D)^{(p-1)}]\tilde{A}_\lambda^D\tilde{B}_\lambda = \mathcal{E}_f = \mathcal{X}_\infty$. *And we also can find that* $\mathcal{D}_\infty = \mathcal{D}_0 = \mathcal{E}_f$.

Chapter 6

Conclusion

In this thesis, we have used the geometric concepts to characterize the solvable space of the triple (E, A, B). And most of the proofs and results are independant of the choice of the basis for the state space. This may bring some benefits in studying geometric control. We have also drawn many conclusions about the (E, A) system under regular assumption, and have combined the geometric properties and numerical solutions. We have also given some examples to illustrate the results. The more exploration of geometric control for (E, A, B) system is left for future study.

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