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由 Ising 向量生成的頂點算子代數及 Griess 代數

Vertex operator algebras and Griess algebras generated by Ising vectors

蘇哲聖

Che-Sheng Su

指導教授：程舜仁 博士

Advisor: Shun-Jen Cheng, Ph.D.

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頂點算子代數(Vertex operator algebra)這個領域，我從一開始未曾接觸過，到現在能夠在其中研究並發表論文，這都要感謝中研院的林正洪老師的指導。一開始林老師教我閱讀《Vertex Operator Algebras and the Monster》[FLM], 讓我知道 lattice vertex operator algebras 的構造方法。接下來教我閱讀一些相關的期刊論文，讓我學到 vertex operator algebras 的公設化定義，了解Virasoro vertex operator algebras 的結構與其 modules 的結構。同時也接觸到Super vertex operator algebras 與其 modules.

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Vertex operator algebras and Griess algebras generated by Ising vectors

Che Sheng Su

(C. S. Su) Department of Mathematics, National Taiwan University, Taipei 10617, Taiwan

E-mail address: d97221001@ntu.edu.tw

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Abstract. In this dissertation, we study Griess algebras generated by Ising vectors. We consider two different cases. In the first case, we study Griess algebras generated by 3 Ising vectors with a common central 2A axial element. In the second case, we consider Griess algebras generated by two $3A$-algebras with a common 3A axial element. In both cases, we classified all possible Griess algebras, up to isomorphism, and related them the McKay's E_7 and E_6 observations about the Baby Monster and the Fischer group.

Key words: vertex operator algebra, Griess algebra, Ising vector, Virasoro algebra.

CHAPTER 1

Introduction

The notion of vertex operator algebras (VOA) is mainly motivated by Frenkel-Lepowsky-Meurman's construction [**FLM**] of the Moonshine module $V^\natural$ conjectured by McKay and Thompson. In addition, vertex operator algebra is also related to conformal field theory (CFT). In fact, the definition of vertex operator algebra is essentially the same as that of chiral algebra in physics literature. Therefore, many algebraic aspects of conformal field theory can be studied by using the representation theory of vertex operator algebras.

By definition, a vertex operator algebra V contains a distinguished element, called the Virasoro (some article called conformal) vector, which makes V into a module of the Virasoro algebra. On the other hand, Frenkel and Zhu [**FZ**] showed that an irreducible highest weight module $L(c,0)$ of the Virasoro algebra of central charge c and highest weight 0 has a natural VOA structure (c.f. Remark 2.16). This VOA is often referred to as simple Virasoro VOA. In [**DMZ**], Dong, Mason and Zhu initiated a study of VOA as a module of simple Virasoro VOA. They showed that the famous Moonshine VOA $V^\natural$ has a full sub-VOA isomorphic to a tensor product of 48-copies of the simple Virasoro VOA $L(1/2,0)$. Partially motivated by [**DMZ**] and Conway's work [**Co**], Miyamoto [**Mi1**] introduced the notion of simple conformal vectors of central charge 1/2, which we call *Ising vectors.* He also developed a method to construct involutions in the automorphism group of a VOA V from Ising vectors. These automorphisms are often called Miyamoto involutions (see Section 2.10 for detail). When V is the famous Moonshine VOA $V^\natural$, Miyamoto [**Mi1**] also showed that there is a $1-1$ correspondence between the $2A$-involutions of the Monster group and the Miyamoto involutions in $V^\natural$ (see also [**Hö**]). This correspondence turns out to be very important for the study of the Monster group. In particular, many mysterious phenomena associated with the $2A$-involutions of the Monster can be interpreted using the theory of VOA [**HLY1, HLY2, LYY1, LYY2, Sa**].

Another important class of VOA is the lattice VOA constructed in [**FLM**]. Given an even positive-definite lattice L, one can construct a VOA $V_L := M(1) \otimes_{\mathbb{C}} \mathbb{C}\{L\}$, where $M(1)$ is an irreducible module of the affine Lie algebra $\hat{\mathfrak{h}}$ with $\mathfrak{h} = \mathbb{C} \otimes_{\mathbb{Z}} L$ and $\mathbb{C}\{L\}$ is a twisted group algebra of L (see Section 3.1). When the lattice is doubly even, i.e., $\langle \alpha, \alpha \rangle \in 4\mathbb{Z}$ for all $\alpha \in L$, the twisted group algebra $\mathbb{C}\{L\}$ is isomorphic to the usual group algebra $\mathbb{C}[L]$ (c.f. (3.2.1)) and the structure will then be much simpler. In particular, we will focus on the doubly even lattice $\sqrt{2}R$ for a root lattice R. In [**DLMN**], many conformal vectors are constructed explicitly in the lattice VOA $V_{\sqrt{2}R}$ when R is a root lattice of A, D, E-type. These conformal vectors are used in [**LYY1, LYY2**] to study McKay's E_8-observation on the Monster simple group. Along with other results, several sub-VOA of the lattice VOA $V_{\sqrt{2}E_8}$ generated by 2 Ising vectors were constructed and studied. There are 9 such sub-VOA. Because of their relations to the 6-transposition property of the Monster group, these VOA are denoted by U_{nX} for $nX \in \{1A, 2A, 2B, 3A, 3C, 4A, 4B, 5A, 6A\}$, where $1A, 2A, 2B, 3A, 3C, 4A, 4B, 5A, 6A$ are the labels for 9 conjugacy classes of the Monster group. We will review the construction and some basic properties of U_{nX} in Chapter 3.

Let $V = \bigoplus_{n=0}^{\infty} V_n$ be a VOA such that $\dim(V_0) = 1$ ($V_0 = \mathrm{Span}\{\mathbb{1}\}$) and $V_1 = 0$. It is well known [**FLM**] that the weight two subspace V_2 has a commutative (non-associative) algebra structure with the product $a \cdot b = a_{(1)}b$ for $a, b \in V_2$. It also has a bilinear form $\langle a, b \rangle \mathbb{1} = a_{(3)}b$. This form is invariant in the sense that $\langle a \cdot b, c \rangle = \langle a, b \cdot c \rangle$ for all $a, b, c \in V_2$. This algebra is often called the Griess algebra of V (c.f. Section 2.7). An element $e \in V_2$ satisfying $e \cdot e = 2e$ is called an *Ising vector* if the sub-VOA generated by e is isomorphic to the simple Virasoro VOA $L(\frac{1}{2}, 0)$ of central charge $\frac{1}{2}$ (c.f. 2.9). In [**Sa**], Griess algebras generated by 2 Ising vectors are classified. He showed that there are exactly 9 Griess algebra structures and they are isomorphic to the Griess algebras $\mathcal{G}U_{nX}$ of the VOA U_{nX}, $nX \in \{1A, 2A, 2B, 3A, 3C, 4A, 4B, 5A, 6A\}$, constructed in [**LYY1, LYY2**]. Based on [**LYY1**] and [**Sa**], one obtains a natural explanation of McKay's E_8-observation using the theory of VOA. We will reviewed Sakuma's work in Chapter 4.

Following the same approach of [**LYY1, LYY2**], McKay's E_7 and E_6-observations on the Baby-Monster and the largest Fischer 3-transposition group Fi_{24} were studied in [**HLY1, HLY2**]. In particular, certain sub-VOA generated by Ising vectors are constructed as a commutant sub-VOA of the lattice VOA $V_{\sqrt{2}E_8}$. In [**HLY1**], certain VOA containing two $2A$-algebras U_{2A} which have a common Ising vector were considered. There are 5 such VOA and they are denoted by $V_{\mathbb{B}(nX)}$, $nX \in \{1A, 2B, 2C, 3A, 4C\}$ (see (3.4.11)), where $1A, 2B, 2C, 3A, 4C$ are 5 conjugacy classes of the Baby Monster group $\mathbb{B}$ (c.f. Section 3.4.2). In addition, VOA containing two $3A$-algebras U_{3A} with a common conformal vector of central charge 4/5 were studied in [**HLY2**]. Three commutant sub-VOA $V_{F(nX)}, nX \in \{1A, 2A, 3A\}$, of $V_{\sqrt{2}E_8}$ were constructed (see (3.4.13)), where $1A, 2A, 3A$ denote 3 conjugacy classes of the Fischer group Fi_{24} (c.f. Section 3.4.3). Motivated by the result of Sakuma [**Sa**], it is natural to ask if the Griess algebras of $V_{F(nX)}$ and $V_{\mathbb{B}(nX)}$ exhaust all possible cases. In Chapter 5 and Chapter 6, we will confirm that the answer is "Yes". In Chapter 5, we will study Griess algebras generated by three Ising vectors e, f, and g such that the sub-VOA generated by e and f and the sub-VOA generated by e and g are both isomorphic to U_{2A}. We will show that there are only 5 possible structures of Griess algebras and they correspond exactly to the Griess algebras $\mathcal{G}V_{\mathbb{B}(nX)}$ of the five VOA $V_{\mathbb{B}(nX)}$, $nX \in \{1A, 2B, 3A, 4B, 2C\}$, constructed in [**HLY1**]. In Chapter 6, we study Griess subalgebras generated by two $3A$-algebras U and U' such that $U \cap U' \cong \mathcal{W}(4/5) = L(4/5, 0) \oplus L(4/5, 3)$. We will show that there are only 3 possibilities, up to isomorphism and they are isomorphic to the Griess algebras of $V_{F(1A)}$, $V_{F(2A)}$, and $V_{F(3A)}$ constructed in [**HLY2**].

The main idea of our classification is to analyze various Griess subalgebras generated by 2 Ising vectors using Sakuma's Theorem and to analyze the symmetric structure from Miyamoto involution τ_e and σ-involution σ_e for an Ising vector e. The invariant condition $\langle a \cdot b, c\rangle = \langle a, b \cdot c\rangle$ is used extensively in our computation. Norton inequality (see Section 2.12) is also essential to our analysis.

The organization is as follows: In Chapter 2, we review some basic definitions and results about vertex operator algebra (VOA). The definition of VOA, modules, dual modules, Ising vectors, Griess algebras, Miyamoto involutions and σ-involutions will be reviewed. Several important results such as Norton inequality will also be recalled. In Chapter 3, we recall the construction of lattice VOA from a even positive-definite lattice. We will specialize it to doubly-even lattices and to the root type lattices $\sqrt{2}E_7$ and $\sqrt{2}E_6$. The construction of the VOA U_{nX}, $V_{\mathbb{B}(nX)}$, and $V_{F(nX)}$ in [**HLY1, HLY2**] will be explained also. In Chapter 4, we recall the result in [**Sa**]. There are exactly 9 structures of Griess algebra generated by 2 Ising vectors. We will describe their basis and their Ising vectors. The product rule, inner product, Miyamoto involutions, and σ-involutions will also be reviewed. In Chapter 5, we classify all Griess subalgebras generated by 2 Griess subalgebras isomorphic to the Griess algebra $\mathcal{G}U_{2A}$ of U_{2A} with a common Ising vector. We show that there are exactly 5 such structures and they are isomorphic to the Griess algebras of $V_{\mathbb{B}(nX)}$. In Chapter 6, we first introduce an order 3 automorphism associated to a conformal vector of central charge 4/5. With the help of the automorphism symmetry, we classify all Griess subalgebras generated by 2 Griess subalgebras isomorphic to the Griess algebra $\mathcal{G}U_{3A}$ of U_{3A} with a common conformal vector of central charge 4/5. We conclude that there are exactly 3 such structures and they are isomorphic to the Griess algebras of $V_{F(nX)}$.

CHAPTER 2

Preliminary

In this chapter, we recall the basic notion of VOA and Griess algebras.

2.1. Formal power series

Let V be a vector space over a field $\mathbb{F}$ of characteristic 0 (usually $\mathbb{F}=\mathbb{C}$ or $\mathbb{R}$). Denote

$$V[[z]] := \{\sum_{i=0}^{\infty} v^i z^i \big| v^i \in V \text{ for all } i\}.$$

$$V[[z,z^{-1}]] := \{\sum_{i\in\mathbb{Z}} v^i z^i \big| v^i \in V \text{ for all } i\}.$$

$$V[[z_1,z_1^{-1},\cdots,z_n,z_n^{-1}]] := \{\sum_{i_1,\cdots,i_n\in\mathbb{Z}} v^{i_1\cdots i_n} z_1^{i_1}\cdots z_n^{i_n} \big| v^{i_1\cdots i_n} \in V \text{ for all } i_1,\cdots,i_n\in\mathbb{Z}\}$$

$$V((z)) := \{\sum_{i=n}^{\infty} v^i z^i \big| n\in\mathbb{Z}, v^i \in V \text{ for all } i\}.$$

Usually, the vector space is an endomorphism ring $\mathrm{End}(V)$.

2.2. Virasoro algebra

Virasoro algebra is an important Lie algebra. It is the central extension of the Lie algebra of the Lie group of small holomorphic motion of a unit circle. Equivalently, Virasoro algebra is a central extension of the Lie algebra

$$\mathrm{Span}_{\mathbb{F}}\{L_n := z^{n+1}\frac{d}{dz} \big| n\in\mathbb{Z}\}.$$

It is well-known that the central extension is unique up to isomorphism.

DEFINITION 2.1. *A* Virasoro algebra Vir *is a Lie algebra with a basis* $\{\mathbf{c}, L_n \big| n \in \mathbb{Z}\}$ *satisfying the bracket rules*

$$[\mathbf{c}, \mathrm{Vir}] = 0,$$
$$[L_m, L_n] = (m-n)L_{m+n} + \delta_{m+n,0}\frac{m^3-m}{12}\mathbf{c}.$$

Moreover, Vir *has the triangular decomposition*

$$\mathrm{Vir} = \mathrm{Vir}^+ \oplus \mathrm{Vir}^0 \oplus \mathrm{Vir}^-,$$

where

$$\mathrm{Vir}^+ := \mathrm{Span}_{\mathbb{F}}\{L_n | \, n \in \mathbb{Z}_{>0}\},\ \mathrm{Vir}^- := \mathrm{Span}_{\mathbb{F}}\{L_n | \, n \in \mathbb{Z}_{<0}\},\ \mathrm{Vir}^0 := \mathbb{F}L_0 \oplus \mathbb{F}\mathbf{c}.$$

2.3. Formal definition of VOA

Next we recall some basic definitions and notations of VOA (c.f. [**FHL**]).

DEFINITION 2.2. *A* vertex operator algebra *(VOA) over a field* $\mathbb{F}$ *is a quadruple* $(V, Y, \mathbb{1}, \omega)$*, satisfying the following conditions:*

(V1) $V = \bigoplus_{n\in\mathbb{Z}} V_n$ *is a* $\mathbb{Z}$*-graded vector space over* $\mathbb{F}$*; for* $v \in V_n$*,* $n = \mathrm{wt}\, v$*, the* weight *of* v*;*

(V2) $\dim V_n < \infty$*;*

(V3) $V_n = 0$ *for* n *sufficiently small;*

(V4) $Y(\cdot, z)$ *is a linear map*

$$\begin{aligned} Y(\cdot,z):\ V &\to (\mathrm{End}_{\mathbb{F}} V)\,[[z, z^{-1}]] \\ v &\to Y(v,z) = \sum_{i\in\mathbb{Z}} v_{(i)}\, z^{-i-1}, \end{aligned}$$

where $\mathrm{End}_{\mathbb{F}}$ *is* $\mathbb{F}$ *linear endmorphism, and* $v_{(i)}$ *(or* $Y(v,z)$*) is called the* vertex operator *associated to* v*;*

(V5) $Y(\mathbb{1}, z) = \mathrm{id}_V$*;*

(V6) $Y(a,z)\mathbb{1} \in V[[z]]$, *and* $\lim_{z\to 0} Y(a,z)\mathbb{1} = a$ *for all* $a \in V$*;*

(V7) $L_i := \omega_{(i+1)}$ *satisfy the Virasoro algebra relations:*

$$[L_i, L_j] = (i-j)L_{i+j} + \delta_{i+j,0}\frac{i^3 - i}{12}\, c,$$

where $c \in \mathbb{F}$ *is some constant, which is called the* rank *of* V*;*

(V8) $L_0 v = (\mathrm{wt}\, v)\, v = nv$ *for* $v \in V_n$*;*

(V9) $Y(L_{-1}v, z) = \dfrac{d}{dz} Y(v,z)$*;*

(V10) *the* Jacobi identity *holds:*

$$\delta(z_1 - z_2, z_0) Y(a, z_1) Y(b, z_2) v - \delta(-z_2 + z_1, z_0) Y(b, z_2) Y(a, z_1) v$$

$$= \delta(z_1 - z_0, z_2) Y(Y(a, z_0)b, z_2) v \in V[[z_0, z_0^{-1}, z_1, z_1^{-1}, z_2, z_2^{-1}]]$$

for all $a, b, v \in V$*, where*

$$\delta(x_1 + x_2, y) := y^{-1} \sum_{i\in\mathbb{Z}} \left(\frac{x_1 + x_2}{y}\right)^i := \sum_{i\in\mathbb{Z}} \sum_{j=0}^{\infty} \binom{i}{j} x_1^{i-j} x_2^j\, y^{-i-1}.$$

The element $\mathbb{1} \in V_0$ *is called the* vacuum element*, and* $\omega \in V_2$ *is called the* Virasoro element*.*

Remark 2.3. Our notation $\delta(x_1 + x_2, y)$ follows that of [**Kac**] and it is the same as the notation $y^{-1}\delta(\frac{x_1+x_2}{y})$ in [**FHL**].

From the definition of VOA, we have the following corollaries.

$$(a_{(m)}b)_{(n)} = \sum_{i\geq 0} \binom{m}{i} (-1)^i \Big(a_{(m-i)} b_{(n+i)} - (-1)^m b_{(m+n-i)} a_{(i)} \Big),$$

$$[a_{(m)}, b_{(n)}] = \sum_{i\geq 0} \binom{m}{i} (a_{(i)}b)_{(m+n-i)},$$

$$(a_{(0)}b)_{(0)} = [a_{(0)}, b_{(0)}],$$

$$a_{(n)}b = \sum_{i\geq 0} \frac{(-1)^{n+i+1}(L_{-1})^i}{i!} b_{(n+i)} a.$$

If $V=\oplus_{i=0}^{\infty}V_i$, $V_0=\mathbb{F}\mathbb{1}$, $V_1=0$, then for $a,b,c\in V_2$,

$$a_{(1)}b=b_{(1)}a,\quad a_{(3)}b=b_{(3)}a,\quad (a_{(1)}b)_{(3)}c=a_{(3)}b_{(1)}c.$$

From the definition, it is easy to show that $a_{(m)}b\in V_{\mathrm{wt}\,a-m-1+\mathrm{wt}\,b}$, and hence we say $\mathrm{wt}\,a_{(m)}=\mathrm{wt}\,a-m-1$. Note also that $Y(a,z)v\in V((z))$ although $Y(a,z)\in(\mathrm{End}_{\mathbb{F}}V)[[z,z^{-1}]]$.

2.4. Module

By the definition of VOA, $v_{(i)}$ acts on V for $v\in V$ and $i\in\mathbb{Z}$. The concept of V-module is similar to modules of rings in usual algebra. A vector space M is a V-module if $v_{(i)}$ acts on M and satisfy the similar properties as $v_{(i)}$ acts on V.

Definition 2.4. *Given a VOA $(V,Y,\mathbb{1},\omega)$, a V-module (M,Y_M) (or briefly (M,Y)) is a $\mathbb{F}$-graded vector space M such that*

(M1) $M=\bigoplus_{n\in\mathbb{F}}M_n$ *is a $\mathbb{F}$-graded vector space over $\mathbb{F}$; for $m\in M_n$, $n=\mathrm{wt}\,m$;*

(M2) $\dim M_n<\infty$*;*

(M3) *For any $h\in\mathbb{F}$, $M_{h+n}=0$ for sufficiently small $n\in\mathbb{Z}$;*

(M4) $Y_M(\cdot,z)$ *(or $Y(\cdot,z)$) is a linear map*

$$\begin{aligned} Y_M(\cdot,z):\ V &\rightarrow (End_{\mathbb{F}}M)\,[[z,z^{-1}]]\\ v &\rightarrow Y_M(v,z)=\sum_{i\in\mathbb{Z}}v_{(i)}\,z^{-i-1};\end{aligned}$$

(M5) $Y_M(\mathbb{1},z)=\mathrm{id}_M$*;*

(M6) *for any $m\in M$ and $v\in V$, $v_{(n)}m=0$ for n sufficiently large;*

(M7) $L_i:=\omega_{(i+1)}\in\mathrm{End}(M)$ *satisfy the Virasoro algebra relations:*

$$[L_i,L_j]=(i-j)L_{i+j}+\delta_{i+j,0}\frac{i^3-i}{12}\,c,$$

where c is the rank (central charge) of V;

(M8) $L_0 m = (\mathrm{wt}\, m)\, m = nm$ *for* $m \in M_n$*;*

(M9) $Y_M(L_{-1}v, z) = \dfrac{d}{dz} Y_M(v, z)$*;*

(M10) *the* Jacobi identity *holds:*

$$\delta(z_1 - z_2, z_0) Y_M(v^1, z_1) Y_M(v^2, z_2) m - \delta(-z_2 + z_1, z_0) Y_M(v^2, z_2) Y_M(v^1, z_1) m$$

$$= \delta(z_1 - z_0, z_2) Y_M(Y(v^1, z_0) v^2, z_2) m \in M[[z_0, z_0^{-1}, z_1, z_1^{-1}, z_2, z_2^{-1}]]$$

for all $v^1, v^2 \in V$ *and* $m \in M$*.*

It is clear from definition that $Y(v, z)m \in M((z))$ for $v \in V, m \in M$.

2.5. Dual module

Definition 2.5. *For* $f(z) \in \mathbb{F}[z, z^{-1}]$*, the operator*

$$f(z)^{L_0} : V[z, z^{-1}] \to V[z, z^{-1}]$$

is defined by $vz^n \to v f(z)^{\mathrm{wt}\, v} z^n$ *for homogeneous* v *and extended linearly. It is well defined since for homogeneous* v*,* $\mathrm{wt}\, v \in \mathbb{Z}_{\geq 0}$ *and* $f(z)^{\mathrm{wt}\, v} \in \mathbb{F}[z, z^{-1}]$*.*

Let $f(z) \in \mathbb{F}[z]$*,* $\varphi \in \mathrm{End}_{\mathbb{F}} V$ *such that for all* $v \in V$*,* $\varphi^n v = 0$ *for* n *large (*n *may depend on* v*). Define*

$$e^{f(z)\varphi} : V[z, z^{-1}] \to V[z, z^{-1}]$$

by $vz^n \to \displaystyle\sum_{i=0}^{\infty} \frac{(\varphi^i v) f(z)^i z^n}{i!}$ *and extended linearly. That is well defined since it is finite sum for each* v*.*

Definition 2.6. *Given a VOA* $(V, Y, \mathbb{1}, \omega)$ *and a* V*-module* (M, Y_M)*, the* contragredient module $(M', Y_{M'})$ *is is defined as*

- $M' = \bigoplus_{n \in \mathbb{F}} M'_n$*, where* $M'_n := (M_n)^* = \mathrm{Hom}_{\mathbb{F}}(M_n, \mathbb{F})$*;*

- *for* $m' \in M'$, $Y_{M'}(v,z)m' \in M'[[z,z^{-1}]]$ *is defined by*

$$\langle Y_{M'}(v,z)m', m\rangle = \langle m', Y_M(e^{zL_1}(-z^{-2})^{L_0}v, z^{-1})m\rangle,$$

which is well defined since $L_1^n v = 0$ *for* n *large and* $e^{zL_1}(-z^{-2})^{L_0}v \in V[z,z^{-1}]$ *(finite sum) by Definition 2.5.*

THEOREM 2.7. *(c.f.* [**FHL**, section 5.2]*) The structure* $(M', Y_{M'})$ *defined in Definition 2.6 is a* V*-module.*

2.6. Morphism

The definition of homomorphism, isomorphism, and automorphism of VOA and of VOA-module are as usual.

DEFINITION 2.8. *Let* $(V, Y, \mathbb{1}, \omega)$ *and* $(\hat{V}, \hat{Y}, \hat{\mathbb{1}}, \hat{\omega})$ *be VOA. A* homomorphism $\phi : V \to \hat{V}$ *is a linear map satisfying* $\phi(\mathbb{1}) = \hat{\mathbb{1}}$, $\phi(\omega) = \hat{\omega}$, *and* $\phi(Y(v^1,z)v^2) = \hat{Y}(\phi(v^1),z)\phi(v^2)$ *for all* v^1, v^2 *in* V. *An isomorphism is a homomorphism which has an inverse homomorphism. An automorphism is an isomorphism from a VOA to itself. We denote the set of all automorphisms of* $(V, Y, \mathbb{1}, \omega)$ *by* $\mathrm{Aut}(V)$.

DEFINITION 2.9. *Let* $(V, Y, \mathbb{1}, \omega)$ *be a VOA.* (M^1, Y_{M^1}) *and* $(M^2, Y_{M^2}, \hat{\mathbb{1}}, \hat{\omega})$ *be* V*-modules. A module* homomorphism *is a linear map* $\phi : M^1 \to M^2$ *(hence extended to a linear map* $M^1[[z,z^{-1}]] \to M^2[[z,z^{-1}]]$*) satisfying* $\phi(Y_{M^1}(v,z)m^1) = Y_{M^2}(v,z)\phi(m^1)$. *An isomorphism is a homomorphism which has an inverse homomorphism. An automorphism is an isomorphism from a module to itself. We denote the set of all automorphisms of* (M, Y_M) *by* $\mathrm{Aut}_V(M)$ *or simply by* $\mathrm{Aut}(M)$.

If (M, Y_M) is isomorphic to $(M', Y_{M'})$ with the isomorphism ϕ, then there is an bilinear form $(\cdot,\cdot)$ on M defined by $(m^1, m^2) = \langle \phi(m^1), m^2\rangle$ satisfying $(Y_M(v,z)m^1, m^2) = (m^1, Y_M(e^{zL_1}(-z^{-2})^{L_0}v, z^{-1})m^2) \in \mathbb{F}((z))$ for m^1, m^2 in M.

2.7. Griess algebra

Definition 2.10. *A bilinear form* $\langle\langle \cdot, \cdot \rangle\rangle$ *on* V *is said to be* invariant (or contragredient, *see* [**FHL**]*) if*

$$\langle\langle Y(a,z)u, v\rangle\rangle = \langle\langle u, Y(e^{zL_1}(-z^{-2})^{L_0}a, z^{-1})v\rangle\rangle \tag{2.7.1}$$

for any a, u, $v \in V$. *That is,* V *as a* V*-module is isomorphic to its dual module.*

Definition 2.11. *A VOA* $V = \bigoplus_{n\in\mathbb{Z}} V_n$ *is said to be of* CFT *type if* $V_n = 0$ *for* $n < 0$ *and* $\dim V_0 = 1$.

The following theorem is proved in [**Li**].

Theorem 2.12. ([**Li**, Theorem 3.1]) *Let* $(V, Y, \mathbb{1}, \omega)$ *be a VOA of CFT type with* $V_1 = 0$. *Then there is a unique symmetric invariant (Definition 2.10) bilinear form* $\langle\langle \cdot, \cdot \rangle\rangle$ *of* $(V, Y, \mathbb{1}, \omega)$ *satisfying* $\langle\langle \mathbb{1}, \mathbb{1} \rangle\rangle = 1$.

Let V be a VOA of CFT type. It is well known [**FLM**] that the weight 1 subspace V_1 has a natural Lie algebra structure defined by

$$[a, b] := a_{(0)}b$$

and has a invariant (in the sense of Lie algebra) symmetry bilinear form given by

$$(a, b)\mathbb{1} = a_{(1)}b.$$

If V_1=0, then it is also well known that the weight 2 subspace V_2 has a commutative (non-associative) algebra structure.

Theorem 2.13 (Theorem 8.9.5 of [**FLM**]). *Let* $(V, Y, \mathbb{1}, \omega)$ *be a VOA of CFT type such that* $V_1 = 0$. *Then the weight 2 space* $\mathcal{G} := V_2$ *has a commutative (non-associative)* *algebra structure defined by the product,*

$$a \cdot b := a_{(1)}b \ (= b_{(1)}a). \tag{2.7.2}$$

Moreover, there is a symmetric bilinear form $\langle \cdot, \cdot \rangle$ *defined by*

$$\langle a, b \rangle \mathbb{1} := a_{(3)}b \ (= b_{(3)}a), \quad a, b \in V_2. \tag{2.7.3}$$

The bilinear form is invariant in the sense that for all a, b, $c \in V_2$,

$$\langle a \cdot b, c \rangle = \langle a, b \cdot c \rangle. \tag{2.7.4}$$

In addition, the bilinear form $\langle \cdot, \cdot \rangle$ *agrees with the invariant form defined in Theorem 2.12, i.e.,* $\langle a, b \rangle = \langle\langle a, b \rangle\rangle$ *for all* a, $b \in V_2$.

DEFINITION 2.14. *The algebra* $\mathcal{G} = \mathcal{G}_V = (V_2, \cdot, \langle \cdot, \cdot \rangle)$ *in Theorem 2.13 is called the* Griess algebra *of* V. *An* automorphism *of* $\mathcal{G}$ *is an automorphism of linear space that preserves the product and the bilinear form. The group of all automorphisms of* $\mathcal{G}$ *is denoted by* $\mathrm{Aut}(\mathcal{G})$. *By Definition 2.8 and Theorem 2.13, it is clear that* $f \in \mathrm{Aut}(V)$ *implies* $f|_{\mathcal{G}} \in \mathrm{Aut}(\mathcal{G})$.

2.8. Virasoro VOA

DEFINITION 2.15. *For constants* c, $h \in \mathbb{F}$, *define an one dimensional* $\mathrm{Vir}^+ \oplus \mathrm{Vir}^0$*-module*

$$\mathbb{F}_{c,h} := \mathbb{F}\mathbf{1}$$

by

$$\mathbf{c} \cdot \mathbf{1} := c\mathbf{1},$$

$$L_0 \cdot \mathbf{1} := h\mathbf{1},$$

and

$$\mathrm{Vir}^+ \cdot \mathbf{1} := 0.$$

Let

$$M(c,h) := \mathrm{Ind}^{\mathrm{Vir}}_{\mathrm{Vir}^+ \oplus \mathrm{Vir}^0} \mathbb{F}_{c,h}.$$

By Poincaré-Birkhoff-Witt Theorem, $M(c,h)$ has basis

$$\{L_{-n_1} \cdots L_{-n_k} \mathbf{1} \big| \, k \in \mathbb{Z}_{\geq 0}, n_1 \geq \cdots \geq n_k \geq 1 \in \mathbb{Z}\}.$$

Let $L(c,h)$ be the irreducible highest weight Vir*-module of central charge c and highest weight h. Then*

$$L(c,h) = M(c,h)/I(c,h),$$

where $I(c,h)$ is the maximal proper sub-module of $M(c,h)$.

Remark 2.16. It is known in [**FZ**, p.163] that the Vir-module $L(c,0)$ has a natural simple VOA structure. This VOA is often called the simple Virasoro VOA of central charge c.

2.9. Ising vectors

Definition 2.17. *Let $(V, Y, \mathbb{1}, \omega)$ be a VOA of CFT type with $V_1 = 0$. An element $e \in V_2$ is called a* conformal vector *with* central charge c.c.$(e) = c \in \mathbb{F}$ *if $\tilde{L}_n := e_{(n+1)}$ satisfy Virasoro algebra relation with central charge c,*

$$[\tilde{L}_m, \tilde{L}_n] = (m-n)\tilde{L}_{m+n} + \delta_{m+n,0}\frac{m^3-m}{12}c.$$

When $\mathrm{Vir}(e) \cong L(c,0)$*, we call e a* simple conformal vector.

Theorem 2.18. (Lemma 5.1, [**Mi1**]) *An element $e \in V_2$ is a conformal vector with central charge c if and only if*

(2.9.1) $$e_{(1)}e = 2e \quad \text{and} \quad e_{(3)}e = \frac{c}{2}\mathbb{1}.$$

Definition 2.19. *For $\mathbb{F} = \mathbb{C}$ or $\mathbb{F} = \mathbb{R}$, a conformal vector e is called an* Ising vector *if* c.c.$(e) = \frac{1}{2}$ *and the sub-VOA* $(\mathrm{Vir}(e), Y, \mathbb{1}, e)$ *generated by e is simple, that is,* $\mathrm{Vir}(e) \cong L(\frac{1}{2}, 0)$.

Remark 2.20. Let $e \in V$ be an Ising vector. Then the sub-VOA $\mathrm{Vir}(e) \cong L(\frac{1}{2}, 0)$ is a rational VOA (i.e., all $\mathrm{Vir}(e)$-modules are completely reducible) and it has exactly 3 irreducible modules $L(\frac{1}{2}, 0)$, $L(\frac{1}{2}, \frac{1}{2})$, and $L(\frac{1}{2}, \frac{1}{16})$ (cf. [**DMZ, Mi1, Mi4**]).

2.10. τ-involution and σ-involution

For an Ising vector $e \in V_2$, one can define a certain involutive automorphism τ_e from e. If $\tau_e = \mathrm{id}$, then one can define another automorphism σ_e.

Definition 2.21. *For a given VOA $(V, Y, \mathbb{1}, \omega)$, an Ising vector $e \in V$, and a constant $h \in \{0, \frac{1}{2}, \frac{1}{16}\}$, let $V_e(h)$ be the sum of all irreducible* $\mathrm{Vir}(e)$*-submodules of V isomorphic to $L(\frac{1}{2}, h)$. Then we have the decomposition (see* [**Mi1**]*)*

$$V = V_e(0) \oplus V_e(\frac{1}{2}) \oplus V_e(\frac{1}{16}).$$

Define a linear map $\tau_e : V \to V$ by

$$\tau_e := \begin{cases} 1 & \text{on } V_e(0) \oplus V_e(\frac{1}{2}), \\ -1 & \text{on } V_e(\frac{1}{16}). \end{cases} \tag{2.10.1}$$

Let V^{τ_e} be the fixed point subspace of τ_e in V, i.e.

$$V^{\tau_e} := \{v \in V \mid \tau_e(v) = v\} = V_e(0) \oplus V_e(\frac{1}{2}). \tag{2.10.2}$$

Define a linear map $\sigma_e : V^{\tau_e} \to V^{\tau_e}$ by

$$\sigma_e := \begin{cases} 1 & \text{on } V_e(0), \\ -1 & \text{on } V_e(\frac{1}{2}). \end{cases} \tag{2.10.3}$$

It was proved in [**Mi1**] *that τ_e and σ_e are automorphisms of V and V^{τ_e}, respectively.*

Theorem 2.22. (Theorem 4.7 and Theorem 4.8 of [**Mi1**]) *Let e be an Ising vector of a VOA V. Then the map τ_e defined in Definition 2.21 is an automorphism of V. Moreover, for any $\rho \in \mathrm{Aut}(V)$, we have $\rho\tau_e\rho^{-1} = \tau_{\rho(e)}$.*

On the fixed point sub-VOA V^{τ_e}, we have $\sigma_e \in \mathrm{Aut}(V^{\tau_e})$. In addition, for any $\varrho \in \mathrm{Aut}(V^{\tau_e})$, we have $\varrho\sigma_e\varrho^{-1} = \sigma_{\varrho(e)}$.

2.11. Eigenspace decomposition

Let V be a VOA of CFT type with $V_1 = 0$ and denote $\mathcal{G} = V_2$.

Let $e \in \mathcal{G}$ be an Ising vector. Then $e_{(1)}$ acts semisimply on the Griess algebra $\mathcal{G}$ and the eigenvalues of $e_{(1)}$ are $0, 2, 1/2$, or $1/16$ only (see Remark 2.20). Since τ and σ involutions are defined via the eigenspace decomposition, we can express the product in the Griess algebra by using these involutions.

Proposition 2.23. (cf. [**Ma, Mi1, Sa**]) *For any Ising vector $e \in \mathcal{G}$, we have an orthogonal decomposition*

$$\mathcal{G} = \mathcal{G}_0^e \oplus \mathcal{G}_2^e \oplus \mathcal{G}_{\frac{1}{2}}^e \oplus \mathcal{G}_{\frac{1}{16}}^e,$$

where $\mathcal{G}_h^e := \{a \in \mathcal{G} | \, e \cdot a = ha\}$. Moreover, $\mathcal{G} \cap V_e(0) = \mathcal{G}_0^e \oplus \mathcal{G}_2^e$, $\mathcal{G} \cap V_e(\frac{1}{2}) = \mathcal{G}_{\frac{1}{2}}^e$, $\mathcal{G} \cap V_e(\frac{1}{16}) = \mathcal{G}_{\frac{1}{16}}^e$.

The next lemma follows immediately from the definitions of τ_e and σ_e.

Lemma 2.24. *Let e be an Ising vector of VOA V. For any $x \in \mathcal{G}$, we have the decomposition $x = x_0 + x_2 + x_{\frac{1}{2}} + x_{\frac{1}{16}}$, where $x_h \in \mathcal{G}_h^e$. Then, $x_{\frac{1}{16}} = \frac{1}{2}\big(x - \tau_e(x)\big)$, $x_{\frac{1}{2}} = \frac{1}{2}\left(\frac{1}{2}(x + \tau_e(x)) - \sigma_e\big(\frac{1}{2}(x + \tau_e(x))\big)\right)$, and $x_2 = 4\langle e, x\rangle e$. Moreover,*

$$e \cdot x = 8\langle e, x\rangle e + \frac{1}{2^2}\left(\frac{1}{2}(x + \tau_e(x)) - \sigma_e\big(\frac{1}{2}(x + \tau_e(x))\big)\right) + \frac{1}{2^5}\big(x - \tau_e(x)\big).$$

If $\tau_e(x) = x$, then we have

$$e \cdot x = 8\langle e, x\rangle e + \frac{1}{2^2}\left(x - \sigma_e(x)\right).$$

2.12. Norton inequality

From now on, we will assume the following condition.

Assumption 1. *Let $(V, Y, \mathbb{1}, \omega)$ be a VOA of CFT type over $\mathbb{R}$. Suppose that $V_1 = 0$ and the invariant bilinear form defined in Theorem 2.12 is positive definite.*

Remark 2.25. Let V be a VOA satisfying Assumption 1. Then the bilinear form $\langle \cdot, \cdot \rangle$ defined on $\mathcal{G} = V_2$ (see Theorem 2.13) is also positive definite. In particular, the Cauchy-Schwartz inequality holds: $\langle a, a\rangle\langle b, b\rangle \geq \langle a, b\rangle^2$, and $\langle a, a\rangle\langle b, b\rangle = \langle a, b\rangle^2$ if and only if a and b are linearly dependent, i.e. $a = rb$ for some $r \in \mathbb{R}$ or $b = 0$. In particular, if a and b are conformal vectors such that $\langle a, b\rangle = \langle a, a\rangle = \langle b, b\rangle$, then $a = b$.

Remark 2.26. Note that the involution τ_e (and σ_e if it is well-defined) also acts on $\mathcal{G} = V_2$ for an Ising vector e.

The next theorem is important to our discussion. The proof can be found in [**Mi1**, Theorem 6.3].

Theorem 2.27. (Norton inequality) *Let V be a VOA satisfying Assumption 1. Then for all a, b in $\mathcal{G} = V_2$, we have*

$$\langle a \cdot a, b \cdot b\rangle \geq \langle a \cdot b, a \cdot b\rangle.$$

In particular, if a, b are idempotents in $\mathcal{G}$, then $\langle a, b\rangle = \langle a \cdot a, b \cdot b\rangle \geq \langle a \cdot b, a \cdot b\rangle \geq 0$.

By Norton inequality, we know that the norm of the product $a \cdot b$ is constrained by the norm of $a \cdot a$ and $b \cdot b$.

CHAPTER 3

Dihedral algebras and McKay's observation

In this chapter, we will recall the construction of the lattice VOA in [**FLM**]. Then we will review McKay's E_8, E_7 and E_6 observations. In addition, the dihedral VOA constructed in [**LYY1, LYY2**] and [**HLY1, HLY2**] will be reviewed.

3.1. Lattice VOA

We will recall the construction of lattice VOA from [**FLM**].

Let $\mathbb{F}$ be a field of characteristic 0. Let $\mathbb{R}^n$ be the $\mathbb{R}$ inner product vector space generated by n orthonormal basis $\{e_1, \cdots, e_n\}$. A lattice in $\mathbb{R}^n$ of rank m is a free $\mathbb{Z}$-module (i.e. free abelian group) generated by m linearly independent elements in $\mathbb{R}^n$ with the restriction positive definite inner product $\langle \cdot, \cdot \rangle$. We say L is a integral lattice if $\langle a, b\rangle \in \mathbb{Z}$ for all a, b in L. We say L is an even lattice if $\langle a, a\rangle \in 2\mathbb{Z}$ for all a in L. Clearly an even lattice is an integral lattice since $\langle a, b\rangle = \frac{1}{2}(\langle a+b, a+b\rangle - \langle a, a\rangle - \langle b, b\rangle)$.

Let $L \subset \mathbb{R}^n$ be a (positive-definite) even lattice of rank m. We will construct a VOA V_L associated to L.

Let $\mathfrak{h}$ be the $\mathbb{F}$ (field of characteristic 0) vector space

$$\mathfrak{h} := L \otimes_{\mathbb{Z}} \mathbb{F}$$

and we view it as an abelian Lie algebra.

Let

$$\hat{\mathfrak{h}} := \mathfrak{h} \otimes \mathbb{C}[\mathbf{t}, \mathbf{t}^{-1}] \oplus \mathbb{F}\mathbf{c}, \quad \text{and} \quad \tilde{\mathfrak{h}} := \hat{\mathfrak{h}} \oplus \mathbb{F}\mathbf{d}$$

be Lie algebras with the bracket defined by

$$[h \otimes \mathbf{t}^n, h' \otimes \mathbf{t}^{n'}] := \delta_{n+n',0}\ n\langle h, h'\rangle \mathbf{c}, \tag{3.1.1}$$

$$[\mathbf{c}, \tilde{\mathfrak{h}}] := 0,$$

$$[\mathbf{d}, h \otimes \mathbf{t}^n] := nh \otimes \mathbf{t}^n,$$

for $h, h' \in \mathfrak{h}, n, n' \in \mathbb{Z}$. We also define a (non-degenerate) bilinear form $\langle \cdot, \cdot \rangle$, on $\tilde{\mathfrak{h}}$ by

$$\langle h \otimes \mathbf{t}^n, h' \otimes \mathbf{t}^{n'}\rangle := \delta_{n,n'}\ \langle h, h'\rangle,$$

$$\langle \mathbf{c}, \hat{\mathfrak{h}}\rangle := 0,$$

$$\langle \mathbf{d}, \hat{\mathfrak{h}}\rangle := 0,$$

$$\langle \mathbf{c}, \mathbf{d}\rangle := 1,$$

$$\langle \mathbf{d}, \mathbf{d}\rangle := 0.$$

It is direct to check that $\tilde{\mathfrak{h}}$ is a Lie algebra and the form $\langle \cdot, \cdot \rangle$ is *invariant* , i.e.

$$\langle [x, y], z\rangle + \langle y, [x, z]\rangle = 0 \quad \text{for all } x, y, z \in \tilde{\mathfrak{h}}.$$

Define the associated Heisenberg algebra,

$$\tilde{\mathfrak{h}}' := [\tilde{\mathfrak{h}}, \tilde{\mathfrak{h}}] = \mathbb{F}\mathbf{c} \oplus \bigoplus_{n \in \mathbb{Z}\setminus\{0\}} \mathfrak{h} \otimes \mathbf{t}^n.$$

We can decompose the Heisenberg algebra as Lie subalgebras

$$\tilde{\mathfrak{h}}' = \tilde{\mathfrak{h}}'^{-} \oplus \mathbb{F}\mathbf{c} \oplus \tilde{\mathfrak{h}}'^{+},$$

where $\tilde{\mathfrak{h}}'^{+} := \bigoplus_{n \in \mathbb{Z}_{>0}} \mathfrak{h} \otimes \mathbf{t}^n$, $\tilde{\mathfrak{h}}'^{-} := \bigoplus_{n \in \mathbb{Z}_{<0}} \mathfrak{h} \otimes \mathbf{t}^n$.

For $\lambda \in \mathbb{F}$, define the one dimensional $\mathbb{F}\mathbf{c} \oplus \tilde{\mathfrak{h}}'^{+}$-module

$$\mathbb{F}_\lambda := \mathbb{F}\mathbf{1}$$

by

$$\mathbf{c1} = \lambda\mathbf{1}, \quad \tilde{\mathfrak{h}}'^{+} \cdot \mathbf{1} = 0.$$

Then we can define the Verma module $M(\lambda)$ to be the induced $\tilde{\mathfrak{h}}'$-module

$$M(\lambda) := \mathrm{Ind}_{\mathbb{F}\mathbf{c}\oplus\tilde{\mathfrak{h}}'^{+}}^{\tilde{\mathfrak{h}}'} \mathbb{F}_\lambda := U(\tilde{\mathfrak{h}}') \otimes_{U(\mathbb{F}\mathbf{c}\oplus\tilde{\mathfrak{h}}'^{+})} \mathbb{F}_\lambda, \tag{3.1.2}$$

where $U(\mathfrak{l})$ is the universal enveloping algebra of the Lie algebra $\mathfrak{l}$. Denote the operator $\alpha \otimes \mathbf{t}^n$ on $M(\lambda)$ by $\alpha(n)$. As vector space, we have

$$M(\lambda) = \mathrm{Span}_{\mathbb{F}}\{\alpha_1(-n_1)\cdots\alpha_k(-n_k)\mathbf{1} \,\big|\, k \in \mathbb{Z}_{\geq 0}, \alpha_i \in \mathfrak{h}, n_i \in \mathbb{Z}_{\geq 1}\ \forall i\}. \tag{3.1.3}$$

Define the *weight* (energy) on $M(\lambda)$ by

$$\mathrm{wt}\,(\alpha_1(-n_1)\cdots\alpha_k(-n_k)\mathbf{1}) := (n_1 + \cdots + n_k).$$

We have $\mathrm{wt}\,(\alpha(n)\cdot\alpha_1(-n_1)\cdots\alpha_k(-n_k)\mathbf{1}) = -n + \mathrm{wt}\,(\alpha_1(-n_1)\cdots\alpha_k(-n_k)\mathbf{1})$, i.e. the weight

$$\mathrm{wt}\,(\alpha(n)) = -n,$$

as an operator, for $n \in \mathbb{Z}$.

Let $\hat{L}$ be a central extension of an even lattice L by the cyclic group $\langle\kappa\rangle$ with $\kappa^2 = 1$ and the commutator map

$$c_0(\alpha, \beta) \equiv \langle\alpha, \beta\rangle \mod 2.$$

That means we have an exact sequence

$$1 \to \langle\kappa\rangle \hookrightarrow (\hat{L}, \cdot) \overset{-}{\twoheadrightarrow} (L, +) \to 0,$$

and

$$aba^{-1}b^{-1} = \kappa^{\langle\bar{a}, \bar{b}\rangle} \tag{3.1.4}$$

for all a, b in $\hat{L}$. Note that such an extension $\hat{L}$ exists and is unique up to isomorphism by [**FLM**, Proposition 5.2.3].

Define the $\mathbb{F}$-algebra

$$\mathbb{F}\{L\} := \mathbb{F}[\hat{L}]/(\kappa+1) = \mathbb{F}[\hat{L}]\big|_{\kappa\to -1}, \tag{3.1.5}$$

where $\mathbb{F}[\hat{L}]$ is the group ring, $(\kappa+1)$ is the (two sided) ideal generated by $\kappa+1$. Let

$$\iota : \mathbb{F}[\hat{L}] \to \mathbb{F}[\hat{L}]/(\kappa+1) = \mathbb{F}\{L\}$$

be the natural projection morphism. Define the *weight* on $\mathbb{F}\{L\}$ by

$$\mathrm{wt}\,(\iota(a)) := \frac{1}{2}\langle \bar{a}, \bar{a}\rangle \in \mathbb{Z}$$

for $a \in \hat{L}$.

Now we can define the lattice VOA V_L. As vector space,

$$V_L := M(1) \otimes_{\mathbb{F}} \mathbb{F}\{L\}, \tag{3.1.6}$$

which is both a left $\tilde{\mathfrak{h}}'$-module and a left $\hat{L}$-module by

$$h(m \otimes \iota(b)) = (hm) \otimes \iota(b),$$

$$a(m \otimes \iota(b)) = m \otimes \iota(ab)$$

for $h \in \tilde{\mathfrak{h}}'$, $a \in \hat{L}$, $m \in M(1)$, $b \in \mathbb{F}\{L\}$. As vector space,

$$V_L = \mathrm{Span}_{\mathbb{F}}\{\alpha_1(-n_1)\cdots\alpha_k(-n_k)\mathbf{1}\otimes\iota(a)\big|\, k \in \mathbb{Z}_{\geq 0}, \alpha_i \in \mathfrak{h}, n_1 \geq \cdots \geq n_k \geq 1 \in \mathbb{Z}, a \in \hat{L}\}.$$

The $\mathbb{Z}$ grading (weight) of V_L comes form the weight of $M(1)$ and of $\mathbb{F}\{L\}$,

$$\mathrm{wt}\,(\alpha_1(-n_1)\cdots\alpha_k(-n_k)\mathbf{1}\otimes\iota(a)) = n_1 + \cdots + n_k + \frac{1}{2}\langle \bar{a}, \bar{a}\rangle. \tag{3.1.7}$$

Then

$$V_L = \bigoplus_{n\in\mathbb{Z}_{\geq 0}} (V_L)_n,$$

where $(V_L)_n := \{v \in V_L \big| \operatorname{wt}(v) = n\}$. Clearly we have

$$(V_L)_0 = \mathbb{F}\mathbb{1}, \tag{3.1.8}$$

where $\mathbb{1} := \mathbf{1} \otimes \iota(1)$.

Extend V_L from $\tilde{\mathfrak{h}}'$-module to $\tilde{\mathfrak{h}}$-module by defining the $\alpha(0)$ and $\mathbf{d}$ action,

$$\alpha(0)(\alpha_1(-n_1)\cdots\alpha_k(-n_k)\mathbf{1} \otimes \iota(a)) := \langle \alpha, \bar{a}\rangle(\alpha_1(-n_1)\cdots\alpha_k(-n_k)\mathbf{1} \otimes \iota(a)), \tag{3.1.9}$$

$$\mathbf{d}(\alpha_1(-n_1)\cdots\alpha_k(-n_k)\mathbf{1} \otimes \iota(a)) = \big(-n_1 - \cdots - n_k - \frac{\langle a, a\rangle}{2}\big)\alpha_1(-n_1)\cdots\alpha_k(-n_k)\mathbf{1} \otimes \iota(a).$$

It is straightforward to check it is well-defined $\tilde{\mathfrak{h}}$-module.

For $\alpha \in \mathfrak{h}$, $\alpha(-1)\mathbf{1} \otimes \iota(1) = \alpha(-1)\mathbb{1} \in V_L$, define

$$Y(\alpha(-1)\mathbb{1}, z) := Y(\alpha, z) := \sum_{n\in\mathbb{Z}} \alpha(n)z^{-n-1}. \tag{3.1.10}$$

For simplicity, we often use $Y(\alpha, z)$ to denote $Y(\alpha(-1)\mathbb{1}, z)$.

Define the normal-ordered product

$$\mathop{\circ}\limits^{\circ}\alpha_1(n_1)\cdots\alpha_k(n_k)\mathop{\circ}\limits^{\circ} := \alpha_{i_1}(n_{i_1})\cdots\alpha_{i_k}(n_{i_k})$$

on $\operatorname{End}(V_L)$ with $\{i_1, \cdots, i_k\} = \{1, \cdots, k\}$ and $n_{i_1} \le \cdots \le n_{i_k}$. Note that $\alpha_1(n_1)\alpha_2(n_2) = \alpha_2(n_2)\alpha_1(n_1)$ unless $n_1 + n_2 = 0$ by (3.1.1). Then we can define $Y(\mathbf{1} \otimes \iota(a), z) \in \operatorname{Hom}_{\mathbb{F}}(V_L, V_L[[z, z^{-1}]]) = (\operatorname{End}_{\mathbb{F}}(V_L))[[z, z^{-1}]]$ by

$$\begin{aligned} Y(\mathbf{1} \otimes \iota(a), z) &= Y(a\mathbb{1}, z) := Y(a, z) && (3.1.11)\\ &:= \mathop{\circ}\limits^{\circ} \exp\Big(\int \big(Y(\bar{a}, z) - \bar{a}_{(0)}z^{-1}\big)dz\Big)\mathop{\circ}\limits^{\circ} az^{\bar{a}} \\ &:= \mathop{\circ}\limits^{\circ} \exp\Big(\sum_{n\in\mathbb{Z}\setminus\{0\}} \frac{\bar{a}(n)}{-n}z^{-n}\Big)\mathop{\circ}\limits^{\circ} az^{\bar{a}} \\ &= \exp\Big(\sum_{n\in\mathbb{Z}_{>0}} \frac{\bar{a}(-n)}{n}z^{n}\Big)\exp\Big(\sum_{n\in\mathbb{Z}_{>0}} \frac{\bar{a}(n)}{-n}z^{-n}\Big)az^{\bar{a}}, && (3.1.12) \end{aligned}$$

where the operator a means left multiplication and the operator $z^{\bar{a}}$ is given by

$$z^{\bar{a}}(\alpha_1(-n_1)\cdots\alpha_k(-n_k)\mathbf{1}\otimes\iota(b)) := \alpha_1(-n_1)\cdots\alpha_k(-n_k)\mathbf{1}\otimes\iota(b)\ z^{\langle\bar{a},\bar{b}\rangle}.$$

Note that (3.1.12) is well-defined since

$$\exp\Big(\sum_{n\in\mathbb{Z}_{>0}}\frac{\bar{a}(n)}{-n}z^{-n}\Big)v$$

is a finite sum for each $v\in V_L$.

We have

$$Y(\mathbb{1},z) = Y(\mathbf{1}\otimes\iota(1),z) = 1\ \ (=\mathrm{id}\,z^0).$$

For $\alpha_1(-n_1)\cdots\alpha_k(-n_k)\mathbf{1}\otimes\iota(a)\in V_L$, define

$$\begin{aligned}Y(\alpha_1(-n_1)\cdots\alpha_k(-n_k)\mathbf{1}\otimes\iota(a),z) :=\ & {}^\circ_\circ\Big(\frac{1}{(n_1-1)!}\big(\frac{d}{dz}\big)^{n_1-1}Y(\alpha_1,z)\Big)\cdots\\ & \Big(\frac{1}{(n_k-1)!}\big(\frac{d}{dz}\big)^{n_k-1}Y(\alpha_k,z)\Big)Y(a,z){}^\circ_\circ,\end{aligned}$$

and extend the definition to $Y(v,z)$ for $v\in V_L$ linearly. Note that this definition is compatible with (3.1.10), (3.1.12).

For $v\in V_L$, define $v_{(n)}\in\mathrm{End}(V_L), n\in\mathbb{Z}$ by

$$Y(v,z) = \sum_{n\in\mathbb{Z}} v_{(n)}z^{-n-1}.$$

Remark 3.1. From (3.1.10) we have

$$(\alpha(-1)\mathbb{1})_{(n)} = \alpha(n),$$

and we can embed $\mathfrak{h}$ into V_L by $\alpha\to\alpha(-1)\mathbb{1}$. Similarly, from (3.1.12) we can embed $\mathbb{F}\{L\}$ into V_L by

$$\iota(a)\to\mathbf{1}\otimes\iota(a).$$

The Virasoro element ω in V_L is defined by

$$\omega := \frac{1}{2}\sum_{i=1}^{m} h_i(-1)h_i(-1)\mathbb{1},$$

where $\{h_1, \cdots h_m\}$ is an orthonormal basis of $\mathfrak{h}$. Note that the definition of ω is independent of the choice of the basis.

Define $L_n := \omega_{(n+1)}$. Then $\mathrm{wt}\,(L_n) = -n$ and $L_0(v) = \mathrm{wt}\,(v)v$ for any homogeneous element v. It is shown in [**FLM**] that $(V_L, Y, \mathbb{1}, \omega)$ is an VOA with the central charge $c = rank(L) = m$.

There is an order 2 automorphism $\theta \in \mathrm{Aut}(V_L)$ (c.f. [**FLM**]) defined by

$$\theta(\alpha_1(-n_1)\cdots\alpha_k(-n_k)\otimes\iota(a)) := (-\alpha_1)(-n_1)\cdots(-\alpha_k)(-n_k)\otimes\iota(a^{-1})(-1)^{\langle\bar{a},\bar{a}\rangle/2}.$$

The fixed space

$$V_L^+ := \{v \in V_L \big|\, \theta(v) = v\}$$

is a sub-VOA. If L is doubly even, then $(V_L)_1 = \mathrm{Span}\{h(-1)\mathbb{1} \mid h \in \mathfrak{h}\}$ and hence we have

(3.1.13) $$(V_L^+)_1 = 0.$$

3.2. $\sqrt{2}$ times root lattices

For an even lattice R, $\sqrt{2}R$ is an doubly even lattice, which means

$$\langle\alpha,\alpha\rangle \in 4\mathbb{Z}$$

for all $\alpha \in \sqrt{2}R$. In this case, $\langle\alpha,\beta\rangle \in 2\mathbb{Z}$ for $\alpha,\beta \in \sqrt{2}R$ and the central extension $\widehat{\sqrt{2}R}$ splits and hence

(3.2.1) $$\mathbb{F}\{\sqrt{2}R\} = \mathbb{F}[\sqrt{2}R] = \bigoplus_{\alpha\in R}\mathbb{F}e^{\sqrt{2}\alpha},$$

where $\{e^{\sqrt{2}\alpha} | \alpha \in R\}$ is the multiplicative abelian group isomorphic to the additive group $\sqrt{2}R$.

In this case, $\mathfrak{h} = (\sqrt{2}R) \otimes_{\mathbb{Z}} \mathbb{F} = R \otimes_{\mathbb{Z}} \mathbb{F}$ if $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$.

The lattice VOA

$$V_{\sqrt{2}R} = \mathrm{Span}_{\mathbb{F}}\{\alpha_1(-n_1)\cdots\alpha_k(-n_k)\mathbf{1}\otimes e^{\sqrt{2}\beta} | k \in \mathbb{Z}_{\geq 0}, \alpha_i \in \mathfrak{h}, n_1 \geq \cdots \geq n_k \geq 1 \in \mathbb{Z}, \beta \in R\}.$$

Suppose $R = R_1 \oplus R_2$ is an orthogonal decomposition. Then $V_{\sqrt{2}R} = V_{\sqrt{2}R_1} \otimes V_{\sqrt{2}R_2}$. If $R' \subset R$ is a sub-lattice, we have a natural inclusion $V_{\sqrt{2}R'} \hookrightarrow V_{\sqrt{2}R}$.

The automorphism $\theta \in \mathrm{Aut}(V_{\sqrt{2}R})$ is given by

$$\theta(\alpha_1(-n_1)\cdots\alpha_k(-n_k) \otimes e^{\sqrt{2}\beta}) := (-\alpha_1)(-n_1)\cdots(-\alpha_k)(-n_k) \otimes e^{-\sqrt{2}\beta}.$$

By (3.1.8) and (3.1.13) we have

$$(V^+_{\sqrt{2}R})_1 = 0, \quad (V^+_{\sqrt{2}R})_0 = \mathbb{F}\mathbb{1}.$$

3.3. $V_{\sqrt{2}R}$ for a root lattice R

If R is generated by its roots, i.e. norm-2 vectors, then the Virasoro element of $V_{\sqrt{2}R}$ is given by

$$\omega_R = \frac{1}{4h} \sum_{\alpha \in \Phi(R)} (\alpha(-1))^2 \mathbb{1},$$

where h is the Coxeter number of R and $\Phi(R)$ is the root system of R. In [**DLMN**], it was shown that the vector defined by

$$\tilde{\omega}_R := \frac{2}{h+2}\omega_R + \frac{1}{h+2} \sum_{\alpha \in \Phi(R)} e^{\sqrt{2}\alpha} \tag{3.3.1}$$

is a conformal vector (cf. Definition 2.17), where $e^x := \mathbf{1} \otimes e^x$ (c.f. Remark 3.1).

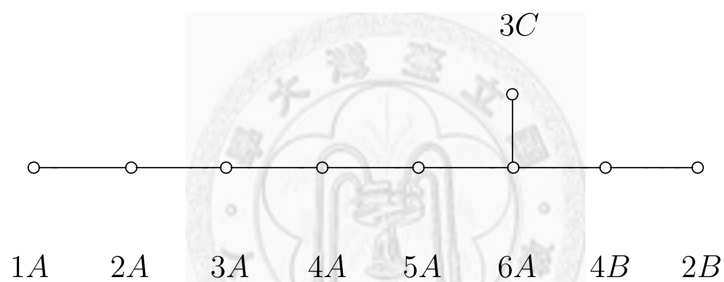
Proposition 3.2. *[c.f. [**DLMN**]] The central charge of $\tilde{\omega}_R$ is $\frac{2n}{n+3}$, 1, $\frac{6}{7}$, $\frac{7}{10}$, and $\frac{1}{2}$ when R is of type A_n, D_n, E_6, E_7, and E_8 respectively .*

Remark 3.3. When $R = E_8$, the conformal vector $\tilde{\omega}_{E_8} \in V_{\sqrt{2}E_8}$ has central charge 1/2 and hence is an Ising vector. The Ising vectors in the lattice VOA V_Λ^+ associated to the Leech lattice Λ are classified in [**LSh**]. It is shown that if $e \in V_\Lambda^+$ is an Ising vector, then there exists a sub-lattice $E < \Lambda$ isomorphic to $\sqrt{2}E_8$ such that $e = \tilde{\omega}_{E_8} \in V_E^+ \subset V_\Lambda^+$.

3.4. McKay's observation

In the late 1970's, John McKay [**McK**] observed that there is an interesting correspondence between the affine E_8 diagram and the 6-transposition property of the Monster group as follows.

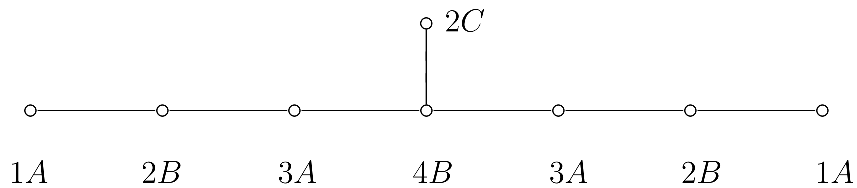
(3.4.1)



It is known that $2A$-involutions of the Monster simple group satisfy a 6-transposition property, that is, $|xy| \leq 6$ (i.e. $(xy)^n = 1$ for some $1 \leq n \leq 6$) for any two $2A$-involutions $x, y \in \mathbb{M}$. In addition, the product xy belongs to one of the following nine conjugacy classes $1A$, $2A$, $3A$, $4A$, $5A$, $6A$, $4B$, $2B$, or $3C$. If we remove the alphabets from the labels, then the nine numbers $1, 2, 3, 4, 5, 6, 4, 2, 3$ are the usual numerical labels of the affine Dynkin E_8-diagram, which are the multiplicities of the corresponding simple roots in the highest root in the E_8 root system. There are similar relations that associate the Baby Monster to the E_7-diagram and Fischer's largest 3-transposition group Fi_{24} to the E_6-diagram as follows.

E_7-observation. Let s, t be 2A-involutions of the Baby Monster. It is known that the product st belongs to one of the Baby Monster conjugacy classes 1A, 2B, 2C, 3A or 4B. McKay noticed [**McK**] that the order of these elements coincide with the numerical labels of the affine E_7 Dynkin diagram and there is a correspondence as below.

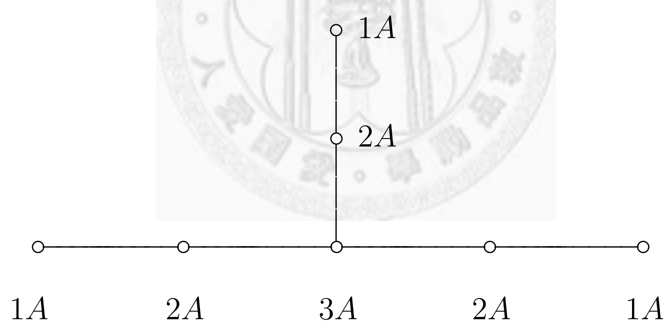
(3.4.2)



In this case, the correspondence is no longer one-to-one but only up to the diagram automorphism.

E_6**-observation.** Similarly, for the Fischer group Fi_{24}, the products of any two 2C-involutions of Fi_{24} belongs to one of the conjugacy classes 1A, 2A or 3A of Fi_{24}. It was again noted by McKay [**McK**] that the order of these elements coincide with the numerical labels of the affine E_6 Dynkin diagram and there is a correspondence as follows:

(3.4.3)



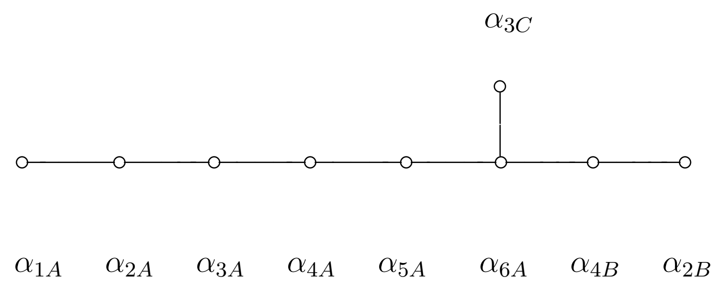
This correspondence is again not one-to-one but only up to diagram automorphisms.

3.4.1. E_8**-case.** In [**LYY1, LYY2**], McKay's E_8 observation has been studied using the VOA $V_{\sqrt{2}E_8}$. Certain VOA generated by 2 Ising vectors were constructed explicitly in $V_{\sqrt{2}E_8}$. There are 9 different cases and these VOA are denoted by U_{1A}, U_{2A}, U_{2B}, U_{3A}, U_{3C}, U_{4A}, U_{4B}, U_{5A}, and U_{6A}. An explanation for McKay's E_8 observation has also been proposed.

Next we will recall the construction of U_{nX} from [**LYY1**]. We assume that $\mathbb{F} = \mathbb{C}$.

For each node $nX \in \{1A, 2A, 3A, 4A, 5A, 6A, 4B, 2B, 3C\}$, we assign a root α_{nX} to the McKay E_8 diagram as follows.

(3.4.4)



That means $\langle \alpha_{nX}, \alpha_{nX} \rangle = 2$ and for $nX \neq mY$, $\langle \alpha_{nX}, \alpha_{mY} \rangle = -1$ if the nodes are connected by an edge and $\langle \alpha_{nX}, \alpha_{mY} \rangle = 0$ otherwise. Then, $\{\alpha_{2A}, \alpha_{3A}, \ldots, \alpha_{3C}\}$ forms a set of simple roots for E_8 and α_{1A} is the negative of the highest root. Moreover,

$$\alpha_{1A} + 2\alpha_{2A} + 3\alpha_{3A} + 4\alpha_{4A} + 5\alpha_{5A} + 6\alpha_{6A} + 4\alpha_{4B} + 2\alpha_{2B} + 3\alpha_{3C} = 0. \tag{3.4.5}$$

For any $nX \in \{1A, 2A, 3A, 4A, 5A, 6A, 4B, 2B, 3C\}$, let $E_8(nX)$ be the sublattice generated by $\{\alpha_{1A}, \ldots, \alpha_{3C}\} \setminus \{\alpha_{nX}\}$. Then $E_8(nX)$ is a rank 8 sublattice of E_8. In fact, $E_8(nX)$ is the root lattice associated with the Dynkin diagram $\hat{E}_8 \setminus \{nX\text{-node}\}$ obtained by removing the corresponding node nX from the affine E_8 diagram. Note that the index $[E_8 : E_8(nX)]$ is equal to n.

The subdiagram $\hat{E}_8 \setminus \{nX\text{-node}\}$ breaks down to several disjoint components. Let $E_8^i(nX) \subset E_8(nX), i = 1, \cdots, \ell$ be the sublattices of E_8 associated to the connected components of $\hat{E}_8 \setminus \{nX\text{-node}\}$. It is clear that the lattice $E_8(nX)$ is an orthogonal sum of $E_8^i(nX)$ for $i = 1, \ldots, \ell$.

Since $[E_8 : E_8(nX)] = n$, we have the decomposition

$$E_8 = \bigcup_{r=0}^{n-1} (r\alpha_{nX} + E_8(nX))$$

The lattice VOA $V_{\sqrt{2}E_8}$ can be decomposed as

$$V_{\sqrt{2}E_8} = \bigoplus_{r=0}^{n-1} V_{\sqrt{2}(r\alpha_{nX}+E_8(nX))},$$

where $V_{\sqrt{2}(r\alpha_{nX}+E_8(nX))} = \bigoplus_{\alpha\in\sqrt{2}(r\alpha_{nX}+E_8(nX))} M(1)\otimes\mathbb{C}e^{\alpha}$ is a $V_{\sqrt{2}E_8(nX)}$-module. The quotient group $E_8/E_8(nX)$ also induces an automorphism ρ_{nX} on $V_{\sqrt{2}E_8}$ defined by

$$\rho_{nX}(u) := \xi_n^r\, u \qquad \text{for } u\in V_{\sqrt{2}(r\alpha_{nX}+E_8(nX))},\ r=0,\cdots,n-1, \tag{3.4.6}$$

where $\xi_n := e^{2\pi i/n}$ is a primitive n-th root of unity.

Let

$$f := \tilde{\omega}_{E_8} = \frac{1}{16}\omega_{E_8} + \frac{1}{32}\sum_{\alpha\in\Phi(E_8)} e^{\sqrt{2}\alpha}$$

be the Ising vector defined as in Section 3.3 (see (3.3.1)) and $f' := \rho_{nX}(\tilde{\omega}_{E_8})$.

Lemma 3.4 (see [**LYY2**]). *As an automorphism of $V_{\sqrt{2}E_8}$, we have*

$$\tau_f\tau_{f'} = \rho_{nX}^{-2} \in \mathrm{Aut}(V_{\sqrt{2}E_8}). \tag{3.4.7}$$

Consider the commutant sub-VOA,

$$U_{nX} := \mathrm{Com}_{V_{\sqrt{2}E_8}}\Big(\mathrm{Vir}\big(\omega_{E_8} - \sum_{i=1}^{\ell}\tilde{\omega}_{E_8^i(nX)}\big)\Big), \tag{3.4.8}$$

where

$$\mathrm{Com}_V(V') := \{v\in V\,|\, v_{(n)}V' = 0 \text{ for all } n\in\mathbb{Z}_{\geq 0}\}$$

denotes the commutant sub-VOA of V' in V. By definition, it is clear that the Virasoro element of U_{nX} is $\sum_{i=1}^{\ell}\tilde{\omega}_{E_8^i(nX)}$. The following result can be found in [**LM, LYY2**] (see also [**GL**]).

Proposition 3.5. *Let U_{nX}, f and f' be defined as above. Then*

(1) *the sub-VOA U_{nX} is generated by f and f';*

(2) *the sub-VOA* U_{nX} *can be embedded into the VOA* $V_\Lambda^+ < V^\natural$ *(the Moonshine VOA). Moreover, the product* $\tau_f\tau_{f'}$ *defines an element in the conjugacy* nX *of* $\mathbb{M}$*.*

Remark 3.6. It is known [**FLM, Mi4**] that the Moonshine VOA $V^\natural$ has a real sub-VOA $V^\natural_{\mathbb{R}}$ such that $V^\natural = V^\natural_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$ and the invariant form on $V^\natural_{\mathbb{R}}$ is positive definite. The VOA U_{nX} also has a positive definite real form $U_{nX,\mathbb{R}}$ [**LYY2**]. Proposition 3.5 still holds if we restrict $V^\natural$ and U_{nX} to their real forms.

3.4.2. E_7**-case.** In [**HLY1**], the $\{3,4\}$-transposition property of the Baby Monster simple group and McKay's E_7-observation were studied. The main idea is to consider a certain commutant sub-VOA in the Moonshine VOA $V^\natural$.

Let e be an Ising vector in $V^\natural$. Then τ_e defines a $2A$-involution of $\mathbb{M}$ and thus $\mathrm{C}_{\mathrm{Aut}(V^\natural)}(\tau_e)$ is a double cover of the Baby Monster simple group $\mathbb{B}$, where $\mathrm{C}_{\mathrm{Aut}(V^\natural)}(\tau_e) := \{g \in \mathrm{Aut}(V^\natural) |\, g\tau_e g^{-1} = \tau_e\}$ is the centralizer of τ_e.

Define

$$V\mathbb{B}^\natural := \mathrm{Com}_{V^\natural}(\mathrm{Vir}(e)),$$

which is called the Baby Monster VOA in [**HLY1**]. Since $\mathrm{C}_{\mathrm{Aut}(V^\natural)}(\tau_e)$ fixes e, it also stabilizes $V\mathbb{B}^\natural$ and hence we have the restriction map $\varphi_e : \mathrm{C}_{\mathrm{Aut}(V^\natural)}(\tau_e) \to \mathrm{Aut}(V\mathbb{B}^\natural)$ such that $\varphi_e(g) := g\big|_{V\mathbb{B}^\natural}$.

In [**Hö**] (see also [**Y**]), it is shown that the automorphism group of $V\mathbb{B}^\natural$, $\mathrm{Aut}(V\mathbb{B}^\natural)$, is isomorphic to the Baby Monster $\mathbb{B}$, and thus we have an exact sequence of groups

$$0 \to \langle \tau_e \rangle \hookrightarrow \mathrm{C}_{\mathrm{Aut}(V^\natural)}(\tau_e) \overset{\varphi_e}{\twoheadrightarrow} \mathrm{Aut}(V\mathbb{B}^\natural) \to 0.$$

For any $2A$-involution $a \in \mathbb{B}$, the inverse image $\varphi_e^{-1}(\langle a \rangle)$ is a Klein's 4-group $\mathbb{Z}_2 \times \mathbb{Z}_2$ such that all involutions belongs to the conjugacy class $2A$. By Miyamoto's correspondence, the group $\varphi_e^{-1}(\langle a \rangle)$ corresponds to a sub-VOA $U < V^\natural$ with $U \cong U_{2A}$ and $e \in U$.

Recall that the $2A$-algebra U_{2A} is isomorphic to

$$L(\frac{1}{2},0)\otimes L(\frac{7}{10},0)\oplus L(\frac{1}{2},\frac{1}{2})\otimes L(\frac{7}{10},\frac{3}{2}),$$

and hence we have

$$\mathrm{Com}_U(\mathrm{Vir}(e))\cong L(\frac{7}{10},0).$$

That means a $2A$-involution of $\mathbb{B}$ determines uniquely a conformal vector of central charge $7/10$ in $V\mathbb{B}^\natural$.

Similar to the case of Ising vectors, one can also define some automorphism with a simple conformal vector of central charge $7/10$. Recall that the simple Virasoro VOA $L(\frac{7}{10},0)$ has 6 inequivant irreducible modules, $L(\frac{7}{10},h)$ with $h=0,\frac{3}{2},\frac{7}{16},\frac{3}{5},\frac{1}{10}$, or $\frac{3}{80}$.

THEOREM 3.7 ([**Mi1**]). *Let V be a VOA and $x\in V$ a simple conformal vector with central charge $7/10$. Denote by $V_x[h]$ the sum of irreducible $\mathrm{Vir}(x)$-submodules isomorphic to $L(\frac{7}{10},h)$ for $h=0,\frac{3}{2},\frac{7}{16},\frac{3}{5},\frac{1}{10}$, or $\frac{3}{80}$. Then the linear map*

$$\tau_x:=\begin{cases}1 & on\ \ V_x[0]\oplus V_x[\frac{3}{2}]\oplus V_x[\frac{3}{5}]\oplus V_x[\frac{1}{10}],\\ -1 & on\ \ V_x[\frac{7}{16}]\oplus V_x[\frac{3}{80}],\end{cases}$$

defines an automorphism of V.

DEFINITION 3.8. *A simple conformal vector u of central charge $7/10$ is said to be of σ-type on V if $V_x[\frac{7}{16}]=V_x[\frac{3}{80}]=0$.*

LEMMA 3.9 ([**Mi1, HLY1**]). *Let $x\in V$ be a simple $c=7/10$ Virasoro vector of σ-type. Then one has the isotypical decomposition*

$$V=V_x[0]\oplus V_x[\frac{3}{2}]\oplus V_x[\frac{1}{10}]\oplus V_x[\frac{3}{5}].$$

Moreover, the linear automorphism $\sigma_x \in \mathrm{End}(V)$ *defined by*

$$(3.4.9) \qquad \sigma_x := \begin{cases} 1 & \text{on } V_x[0] \oplus V_x[\frac{3}{5}], \\ \\ -1 & \text{on } V_x[\frac{3}{2}] \oplus V_x[\frac{1}{10}] \end{cases}$$

is an automorphism of V.

Remark 3.10. In [**HLY1**], it is shown that there is a 1-1 corrrespondence between the 2A-involutions of the Baby Monster group and the simple $c = 7/10$ conformal vectors of σ-type the Baby Monster VOA $V\mathbb{B}^{\natural}$.

3.4.2.1. *Commutant subalgebras* $U_{\mathbb{B}(nX)}$ *and* $V_{\mathbb{B}(nX)}$. Next we will recall the construction of certain conformal vectors of central charge 7/10 and the commutant subalgebras $U_{\mathbb{B}(nX)}$ and $V_{\mathbb{B}(nX)}$ from [**HLY1**]. Similar to the E_8 case, we will first define an automorphism $\rho_{\mathbb{B}(nX)} \in \mathrm{Aut}(V_{\sqrt{2}E_7})$.

For each node nX, $nX \in \{1A, 2B, 3A, 4B, 2C\}$, of the McKay E_7-diagram (cf. (3.4.2)), let $E_7(nX) < E_7$ be the root sublattice associated with the Dynkin diagram $\hat{E}_7 \setminus \{nX-$ node$\}$ obtained by removing the corresponding node nX. We also denote the simple root associated to the node nX by β_{nX}. Then $E_7 = \bigcup_{r=0}^{n-1}(r\beta_{nX} + E_7(nX))$ and

$$V_{\sqrt{2}E_7} = \bigoplus_{r=0}^{n-1} V_{\sqrt{2}(r\beta_{nX}+E_7(nX))}.$$

The automorphism $\rho_{\mathbb{B}(nX)} : V_{\sqrt{2}E_7} \to V_{\sqrt{2}E_7}$ is defined by

$$\rho_{\mathbb{B}(nX)}(u) = \xi_n^r u \qquad \text{for } u \in V_{\sqrt{2}(r\beta_{nX}+E_7(nX))},\ r = 0, \cdots, n-1.$$

In fact

$$(3.4.10) \qquad \rho_{\mathbb{B}(nX)} = \exp\Big(\frac{2\pi i}{n}\delta_{nX}(0)\Big)$$

for some $\delta_{nX} \in (\sqrt{2}E_7(nX))^* \subset \mathfrak{h}$ (c.f. (3.1.9)) as an automorphism (see [**HLY1, LYY1**]).

REMARK 3.11. Strictly speaking, the root β_{nX} is not well defined since there are more than one nodes with the label nX. However, the isometry type of $E_7(nX)$ and the conjugacy class of the automorphism $\rho_{\mathbb{B}(nX)}$ are uniquely determined by the label nX.

NOTATION 3.12. *Let* $\tilde{f} := \tilde{\omega}_{E_7}$ *be the conformal vector defined as in* (3.3.1) *with* $R = E_7$ *and* $\tilde{g} := \rho_{\mathbb{B}(nX)}(f)$. *Then* $\tilde{f}$ *and* $\tilde{g}$ *are conformal vectors of central charge* $7/10$.

Similar to the E_8 case, we denote the sublattices associated to connected components of $\hat{E}_7 \setminus \{nX\text{-node}\}$ by $E_7^i(nX)$, $i = 1, \dots, \ell$ and define the commutant sub-VOA

$$U_{\mathbb{B}(nX)} := \mathrm{Com}_{V_{\sqrt{2}E_7}}\Big(\mathrm{Vir}\big(\omega_{E_7} - \sum_{i=1}^{\ell} \tilde{\omega}_{E_7^i(nX)}\big)\Big).$$

In [**HLY1**], it is shown that $\tilde{f}, \tilde{g}$ are contained in $U_{\mathbb{B}(nX)}$ but in general, the VOA (or the Griess subalgebra) generated by $\tilde{f}$ and $\tilde{g}$ is not equal to $U_{\mathbb{B}(nX)}$ (or the Griess algebra of $U_{\mathbb{B}(nX)}$). Therefore, we will consider some bigger sub-VOA.

First we fix an embedding of E_7 into E_8. Then

$$\mathrm{Ann}_{E_8}(E_7) := \{\alpha \in E_8 |\, \langle \alpha, E_7 \rangle = 0\} \cong A_1$$

and we obtain an embedding of $A_1 \oplus E_7$ into E_8. Note that such an embedding is unique up to an automorphism of E_8.

Now define

$$V_{\mathbb{B}(nX)} := \mathrm{Com}_{V_{\sqrt{2}E_8}}\Big(\mathrm{Vir}\big(\omega_{E_8} - \tilde{\omega}_{\mathrm{Ann}_{E_8}(E_7)} - \sum_{i=1}^{\ell} \tilde{\omega}_{E_7^i(nX)}\big)\Big). \tag{3.4.11}$$

Then the Virasoro element of $V_{\mathbb{B}(nX)}$ is $\tilde{\omega}_{\mathrm{Ann}_{E_8}(E_7)} + \sum_{i=1}^{\ell} \tilde{\omega}_{E_7^i(nX)}$ and by definition, it is clear that

$$U_{\mathbb{B}(nX)} \cong \mathrm{Com}_{V_{\mathbb{B}(nX)}}\Big(\mathrm{Vir}\big(\tilde{\omega}_{\mathrm{Ann}_{E_8}(E_7)}\big)\Big).$$

Since $\mathrm{Ann}_{E_8}(E_7) \cong A_1$, the central charge of $\tilde{\omega}_{\mathrm{Ann}_{E_8}(E_7)}$ is $\frac{1}{2}$ by Proposition 3.2.

Notation 3.13. *Let*

$$e := \tilde{\omega}_{\mathrm{Ann}_{E_8}(E_7)} \quad \text{and} \quad f = \tilde{\omega}_{E_8}.$$

Then both e and f are Ising vectors in $V_{\mathbb{B}(nX)}$ (c.f. [**LYY1, LYY2**]*). By* (3.4.10)*, we have $\rho_{\mathbb{B}(nX)} = \exp(\frac{2\pi i}{n}\delta_{nX}(0))$. It also defines an automorphism of $V_{\sqrt{2}E_8}$ by the embedding of E_7 to E_8. Define*

$$g := \rho_{\mathbb{B}(nX)}(f).$$

Then g is also an Ising vector. Moreover, $\mathrm{VOA}(e,f) \cong \mathrm{VOA}(e,g) \cong U_{2A}$.

The following results are proved in [**HLY1**].

Proposition 3.14. *Let e, f, g be defined as in Notation 3.13 and let $\tilde{f}$, $\tilde{g}$ be defined as in Notation 3.12. Then*

(1) $\mathrm{Com}_{\mathrm{VOA}(e,f)}(\mathrm{Vir(e)}) = Vir(\tilde{f})$ *and* $\mathrm{Com}_{\mathrm{VOA}(e,g)}(\mathrm{Vir(e)}) = Vir(\tilde{g})$*;*

(2) *the VOA $V_{\mathbb{B}(nX)}$ can be embedded into $V_\Lambda^+ < V^\natural$ and $U_{\mathbb{B}(nX)}$ can be embedded into $V\mathbb{B}^\natural$ for any $nX = 1A, 2B, 3A, 4B, 2C$. Moreover, $\varphi_e(\tau_f\tau_g) = \sigma_{\tilde{f}}\sigma_{\tilde{g}}$ belongs to the conjugacy class nX of the Baby Monster.*

Remark 3.15. As in the E_8 case, we can also consider the (positive definite) real forms of $V\mathbb{B}^\natural$, $V_{\mathbb{B}(nX)}$, etc. The conclusion in Proposition 3.14 will still hold.

In Chapter 5, we will study Griess-algebras generated by three Ising vectors e, f, and g such that the sub-VOA generated by e and f and the sub-VOA generated by e and g are both isomorphic to U_{2A}. We say that such a configuration is of *central* $2A$-*type*. Under this assumption, we will show that there are only 5 possible structures of sub-Griess-algebras and they correspond exactly to the Griess algebras of the five VOA $V_{\mathbb{B}(nX)}$, $nX \in \{1A, 2B, 3A, 4B, 2C\}$.

3.4.3. $R = E_6$ case. The above method can also be used to study McKay's E_6-observation [**HLY2**].

For each node nX, $nX \in \{1A, 2A, 3A\}$, of the E_6-diagram (cf. (3.4.3)), let $E_6(nX) < E_6$ be the root sublattice associated with the Dynkin diagram $\hat{E}_6 \setminus \{nX\text{-node}\}$ obtained by removing the corresponding node nX. We also use γ_{nX} to denote the simple root associated to the node nX. Then $E_6 = \bigcup_{r=0}^{n-1}(r\gamma_{nX} + E_6(nX))$ and

$$V_{\sqrt{2}E_6} = \bigoplus_{r=0}^{n-1} V_{\sqrt{2}(r\gamma_{nX}+E_6(nX))}.$$

We also obtain an automorphism $\rho_{F(nX)} : V_{\sqrt{2}E_6} \to V_{\sqrt{2}E_6}$ defined by

$$\rho_{F(nX)}(u) = \xi_n^r\, u \qquad \text{for } u \in V_{\sqrt{2}(r\gamma_{nX}+E_6(nX))},\ r = 0, \cdots, n-1.$$

Note that

$$\rho_{F(nX)} = \exp(\frac{2\pi i}{n}\delta'_{nX}(0)) \tag{3.4.12}$$

for some $\delta'_{nX} \in (\sqrt{2}E_6(nX))^*$ as an automorphism [**HLY2, LYY1**].

Notation 3.16. *Let $u := \tilde{\omega}_{E_6}$ be the conformal vector defined as in* (3.3.1) *and $u' := \rho_{F(nX)}(u)$. Then u and u' are conformal vectors of central charge* $6/7$.

As in the E_8 and E_7 cases, we use $E_6^i(nX)$ to denote the sublattice associated to the connected components of the Dynkin diagram $\hat{E}_6 \setminus \{nX\text{-node}\}$. We also define the commutant sub-VOA

$$U_{F(nX)} := \mathrm{Com}_{V_{\sqrt{2}E_6}}\Big(\mathrm{Vir}\big(\omega_{E_6} - \sum_{i=1}^{\ell}\tilde{\omega}_{E_6^i(nX)}\big)\Big).$$

Fix an embedding of E_6 into E_8. Then $\mathrm{Ann}_{E_8}(E_6) := \{\alpha \in E_8 |\, \langle \alpha, E_6\rangle = 0\} \cong A_2$ and we obtain an embedding of $A_2 \oplus E_6$ into E_8.

Similar to the E_7 case, we also consider the commutant sub-VOA

$$V_{F(nX)} := \mathrm{Com}_{V_{\sqrt{2}E_8}}\Big(\mathrm{Vir}\big(\omega_{E_8} - \tilde{\omega}_{\mathrm{Ann}_{E_8}(E_6)} - \sum_{i=1}^{\ell} \tilde{\omega}_{E_6^i(nX)}\big)\Big). \tag{3.4.13}$$

Since $\mathrm{Ann}_{E_8}(E_6) \cong A_2$, the central charge of $\tilde{\omega}_{\mathrm{Ann}_{E_8}(E_6)}$ is 4/5 by Proposition 3.2. Moreover, $U_{F(nX)} = \mathrm{Com}_{V_{F(nX)}}\Big(\mathrm{Vir}\big(\tilde{\omega}_{\mathrm{Ann}_{E_8}(E_6)}\big)\Big)$.

From now on, set $\mu := \tilde{\omega}_{\mathrm{Ann}_{E_8}(E_6)} = \tilde{\omega}_{A_2}$. Recall that the quotient group $E_8/(A_2 \oplus E_6)$ induces an automorphism $\rho := \rho_{3A} \in \mathrm{Aut}(V_{\sqrt{2}E_8})$ (cf. (3.4.6)). Let

$$a_0 := \tilde{\omega}_{E_8} \qquad \text{and} \qquad a_1 := \rho(a_0).$$

Then both a_0, a_1 are Ising vectors in $V_{F(nX)}$ (c.f. [**LYY1, LYY2**]) and the sub-VOA $\mathrm{VOA}(a_0, a_1)$ generated by a_0, a_1 is isomorphic to U_{3A}. Moreover, $\mu = \tilde{\omega}_{A_2} \in \mathrm{VOA}(a_0, a_1)$ and is fixed by $\tau_{a_0}\tau_{a_1}$ (see [**LYY2**]).

By (3.4.12), the map $\rho_{F(nX)} = \exp(\frac{2\pi i}{n}\delta'_{nX}(0))$ also defines an automorphism on $V_{\sqrt{2}E_8}$. Define

$$b_0 = \rho_{F(nX)}(a_0) \qquad \text{and} \qquad b_1 = \rho_{F(nX)}(a_1).$$

Then b_0, b_1 are also Ising vectors and they generate a 3A-algebra in $V_{\sqrt{2}E_8}$.

Since $\rho_{F(nX)}$ fixes $V_{\sqrt{2}\mathrm{Ann}_{E_8}(E_6)}$ pointwisely, it fixes the conformal vector μ and the sub-VOA $\mathrm{Com}_{V_{\sqrt{2}A_2}}(\mathrm{Vir}(\omega_{A_2} - \mu))$, which is isomorphic to the W_3-algebra $\mathcal{W}(4/5) \cong L(4/5, 0) \oplus L(4/5, 3)$ (cf. [**HLY2, SY**]).

The next result can be found in [**HLY2**].

PROPOSITION 3.17. *Let* $U = \mathrm{VOA}(a_0, a_1)$ *and* $U' = \mathrm{VOA}(b_0, b_1)$. *Then*

(1) *the VOA* $V_{F(nX)}$ *is generated by* U *and* U' *and*

$$U \cap U' \cong \mathcal{W}(\frac{4}{5}) \cong L(\frac{4}{5}, 0) \oplus L(\frac{4}{5}, 3).$$

(2) *u is the Virasoro elemnent of* $\mathrm{Com}_U(\mathrm{Vir}(\mu))$ *and u' is the Virasoro elemnent of* $\mathrm{Com}_{U'}(\mathrm{Vir}(\mu))$*, where u and u' are defined as in Notation 3.16.*

3.4.3.1. *The VOA $V\!F^\natural$.* Next we will recall the properties of a commutant sub-VOA $V\!F^\natural$ from [**HLY2**].

Let g be a 3A-element of the Monster $\mathbb{M}$. Then the normalizer $N_{\mathbb{M}}(\langle g\rangle)$ is isomorphic to $3.\mathrm{Fi}_{24}$ and acts on $V^\natural$. A character theoretical consideration in [**Co, MeN**] indicates that the centralizer $C_{\mathbb{M}}(g) \cong 3.\mathrm{Fi}_{24}'$ fixes a unique simple conformal vector μ of central charge $4/5$ in $V^\natural$, where $\mathrm{Fi}_{24}' := \langle aba^{-1}b^{-1} | a, b \in \mathrm{Fi}_{24}\rangle$ is the derived subgroup. In fact, it was also shown that $C_{\mathbb{M}}(g)$ actually fixes an extension $\mathcal{W} \cong \mathcal{W}(\frac{4}{5}) = L(\frac{4}{5}, 0) \oplus L(\frac{4}{5}, 3)$ of $\mathrm{Vir}(\mu)$ in $V^\natural$.

Definition 3.18. *Define the commutant sub-VOA*

$$V\!F^\natural := \mathrm{Com}_{V^\natural}(\mathcal{W}) = \mathrm{Com}_{V^\natural}(\mathrm{Vir}(\mu)).$$

The VOA $V\!F^\natural$ is called the Fischer group VOA in [**HLY2**].

A simple observation shows that $N_{\mathbb{M}}(\langle g\rangle)$ acts naturally on $V\!F^\natural = \mathrm{Com}_{V^\natural}(\mathcal{W})$. In fact, the Fischer group Fi_{24} can be realized as a subgroup of $\mathrm{Aut}(V\!F^\natural)$.

Theorem 3.19 ([**HLY2**]). *Let $\varphi_\mu : N_{\mathbb{M}}(\langle g\rangle) \to \mathrm{Aut}(V\!F^\natural)$ be the natural restriction map. Then the image of φ_μ is isomophic to* Fi_{24}*. Therefore, the automorphism group* $\mathrm{Aut}(V\!F^\natural)$ *of $V\!F^\natural$ contains* Fi_{24} *as a subgroup. Moreover, let $\mathcal{X}$ be the full-subalgebra of $V\!F^\natural$ generated by its weight 2 subspace. Then* $\mathrm{Aut}(\mathcal{X}) \simeq \mathrm{Fi}_{24}$*.*

By the theorem above, we have an exact sequence

$$0 \to \langle g\rangle \to \mathrm{N}_{\mathrm{Aut}(V^\natural)}(\langle g\rangle) \to \mathrm{Fi}_{24} \to 0.$$

Let t be a $2C$-involution of Fi_{24}. Then the inverse image $\varphi_\mu^{-1}(\langle t\rangle)$ is isomorphic to the symmetry group S_3 and is generated by two $2A$-involutions of the Monster [**ATLAS**]. Therefore, $\varphi_\mu^{-1}(\langle t\rangle)$ corresponds to a $3A$-subalgebra U in $V^\natural$ with $\mu \in U$ by the Miyamoto correspondence. In addition, we have

$$\mathrm{Com}_U(\mathrm{Vir}(\mu)) \cong L(\frac{6}{7},0) \oplus L(\frac{6}{7},5)$$

(see [**HLY2, LYY2, SY**]). In other words, a $2C$-involution of Fi_{24} determines an extended Virasoro VOA $L(\frac{6}{7},0) \oplus L(\frac{6}{7},5)$ in the Fischer group VOA $V\!F^\natural$.

DEFINITION 3.20. *A simple conformal vector $u \in V$ of central charge $6/7$ is said to be of σ-type in V if $V_u[h] = 0$ unless $h = 0,\ 5,\ 1/7,\ 5/7,\ 12/7,\ 22/7$. The subspace $V_u[h]$ is defined to be the sum of all irreducible* Vir(u)*-modules of V isomorphic to $L(6/7, h)$.*

LEMMA 3.21. *Let $u \in V$ be a simple $c = 6/7$ conformal vector of σ-type. Then the linear map σ_u given by*

$$\sigma_u := \begin{cases} 1 & \text{on } V_u[0] \oplus V_u[\frac{5}{7}] \oplus V_u[\frac{22}{7}], \\ -1 & \text{on } V_u[5] \oplus V_u[\frac{12}{7}] \oplus V_u[\frac{1}{7}]. \end{cases} \tag{3.4.14}$$

is an automorphism of V.

The next theorem is also proved in [**HLY2**].

PROPOSITION 3.22 (Proposition 5.15 and Theorem 5.16 of [**HLY2**]). *For any $nX = 1A$, $2A$ or $3A$, the VOA $V_{F(nX)}$ can be embedded into the Moonshine VOA $V^\natural$ and the VOA $U_{F(nX)}$ can be embedded into $V\!F^\natural = \mathrm{Com}_{V^\natural}(\mathrm{Vir}(\mu))$. Moreover, $\sigma_u\sigma_{u'} = \varphi_\mu(\tau_{a_1}\tau_{b_1})$ defines an element of the conjugacy nX in* Fi_{24}.

In Chapter 6, we will study Griess-algebras generated by two $3A$-algebras U and U' such that their intersection contains a sub-VOA isomorphic to $\mathcal{W}(4/5)$. We will show that there are only 3 possibilities, up to isomorphism and they are isomorphic to the Griess algebras associated to $V_{F(1A)}$, $V_{F(2A)}$, and $V_{F(3A)}$.

CHAPTER 4

Griess algebra generated by 2 Ising vectors

From now on, all VOA and their Griess algebras are over $\mathbb{R}$ unless otherwise stated. We also assume every VOA satisfies Assumption 1, i.e., a VOA of CFT type with $V_1 = 0$ and the invariant form on V is positive definite.

In [**Sa**], Griess algebras generated by 2 Ising vectors in a VOA satisfying Assumption 1 are classified. There are 9 cases and the structures of these Griess algebras are determined (see also [**IPSS**, Table 3]).

NOTATION 4.1. *For g_1, $g_2 \in \mathrm{Aut}(\mathcal{G})$, define $\mathrm{Gp}\langle g_1, g_2\rangle$ to be the subgroup generated by g_1 and g_2. For $g \in \mathrm{Aut}(\mathcal{G})$, $S \subset \mathcal{G}$, define $g \cdot S$ to be the subset $\{g(x) | \, x \in S\} \subset \mathcal{G}$. For any $G < \mathrm{Aut}(\mathcal{G})$ and $S \subset \mathcal{G}$, set $G \cdot S := \{g(x) \mid x \in S, g \in G\} \subset \mathcal{G}$.*

NOTATION 4.2. *Let V be a VOA satisfying Assumption 1. Let x_0, x_1 be Ising vectors in V_2. Let $D := \mathrm{Gp}\langle \tau_{x_0}, \tau_{x_1}\rangle$ be the dihedral group generated by τ_{x_0}, τ_{x_1} and $\rho := \tau_{x_1}\tau_{x_0}$. Set $I_0 = D \cdot x_0$, $I_1 = D \cdot x_1$ (Notation 4.2), and $I = I_0 \cup I_1$.*

LEMMA 4.3. *(cf.* [**Sa**, Lemma 4.1, 4.2]*) Let V, x_0, x_1, I_0, I_1 and I be defined as in Notation 4.2. Then*

(1) $|I_0| = |I_1|$;

(2) $I_0 = I_1$ *if and only if* $n = |I_0|$ *is odd. In this case,* $x_1 = \rho^{(n+1)/2}(x_0)$;

(3) $|I| \leq 6$ *and* $(\tau_{x_0}\tau_{x_1})^{|I|} = 1$ *as an automorphism of* V.

THEOREM 4.4. *[cf.* [**Sa**] *and* [**IPSS**]*] Let V be a VOA satisfying Assumption 1. Let x_0, x_1 be Ising vectors in V_2 and let I_0, I_1 and I be defined as in Notation 4.2. Then* *the Griess subalgebra $\mathcal{G}$ generated by x_0 and x_1 in $\mathcal{G}_V = V_2$ is isomorphic to one of the following 9 algebras: $\mathcal{G}U_{1A}$, $\mathcal{G}U_{2A}$, $\mathcal{G}U_{2B}$, $\mathcal{G}U_{3A}$, $\mathcal{G}U_{3C}$, $\mathcal{G}U_{4A}$, $\mathcal{G}U_{4B}$, $\mathcal{G}U_{5A}$, and $\mathcal{G}U_{6A}$. Moreover, $I = I_0 \cup I_1$ is the set of all Ising vectors in $\mathcal{G}$ unless $\mathcal{G} \cong \mathcal{G}U_{2A}$, $\mathcal{G}U_{4B}$, or $\mathcal{G}U_{6A}$. If $\mathcal{G} \cong \mathcal{G}U_{2A}, \mathcal{G}U_{4B}$, or $\mathcal{G}U_{6A}$, then the number of Ising vectors in $\mathcal{G}$ is equal to $|I_0 \cup I_1| + 1$.*

The structures of the 9 algebras can be summarized as follows.

$\mathcal{G}\{x_0, x_1\}$	$\mathcal{G}U_{1A}$	$\mathcal{G}U_{2A}$	$\mathcal{G}U_{2B}$	$\mathcal{G}U_{3A}$	$\mathcal{G}U_{3C}$	$\mathcal{G}U_{4A}$	$\mathcal{G}U_{4B}$	$\mathcal{G}U_{5A}$	$\mathcal{G}U_{6A}$
$\langle x_0, x_1 \rangle$	$\frac{1}{2^2}$	$\frac{1}{2^5}$	0	$\frac{13}{2^{10}}$	$\frac{1}{2^8}$	$\frac{1}{2^7}$	$\frac{1}{2^8}$	$\frac{3}{2^9}$	$\frac{5}{2^{10}}$

4.0.4. $\mathcal{G}U_{1A}$. *In this case, $x_0 = x_1$, and hence $\mathcal{G} = \mathrm{Span}_{\mathbb{R}}\{x_0\}$ and $\dim \mathcal{G} = 1$. Therefore, $I = I_0 = I_1 = \{x_0\}$. The multiplication and the bilinear form are given by $x_0 \cdot x_0 = 2x_0$ and $\langle x_0, x_0 \rangle = \frac{1}{2^2}$.*

4.0.5. $\mathcal{G}U_{2A}$. *In this case, $\tau_{x_0}(x_1) = x_1$, $\tau_{x_1}(x_0) = x_0$, $\langle x_0, x_1 \rangle \neq 0$. Let $x_2 := \sigma_{x_0}(x_1)$. Then $\mathcal{G} = \mathrm{Span}_{\mathbb{R}}\{x_0, x_1, x_2\}$ and $\dim \mathcal{G} = 3$. In addition, $I_0 = \{x_0\}$, $I_1 = \{x_1\}$ and there are 3 Ising vectors in $\mathcal{G}$. The multiplication and the bilinear form are given by*

$$x_i \cdot x_j = \frac{1}{2^2}(x_i + x_j - x_k) \quad \text{and} \quad \langle x_i, x_j \rangle = \frac{1}{2^5} \quad \text{for } \{i, j, k\} = \{0, 1, 2\}. \tag{4.0.1}$$

Note also that $\tau_{x_i} = \mathrm{id}$ on $\mathcal{G}$ and $\sigma_{x_i}(x_j) = x_k$ for $\{i, j, k\} = \{0, 1, 2\}$. We call the ordered set (x_0, x_1, x_2) a normal $\mathcal{G}U_{2A}$ basis.

4.0.6. $\mathcal{G}U_{2B}$. *In this case, $\tau_{x_0}(x_1) = x_1$, $\tau_{x_1}(x_0) = x_0$, and $\langle x_0, x_1 \rangle = 0$. Then $\mathcal{G} = \mathrm{Span}_{\mathbb{R}}\{x_0, x_1\}$ and $\dim \mathcal{G} = 2$. In addition, $I_0 = \{x_0\}$, $I_1 = \{x_1\}$, and there are exactly 2 Ising vectors in $\mathcal{G}$. The multiplication and the bilinear form are given by*

$$x_i \cdot x_j = 0 \quad \text{and} \quad \langle x_i, x_j \rangle = 0 \quad \text{for } i \neq j. \tag{4.0.2}$$

Note that both τ_{x_i} and σ_{x_i} act trivially on $\mathcal{G}$. We call (x_0, x_1) a normal $\mathcal{G}U_{2B}$ basis.

4.0.7. $\mathcal{G}U_{3A}$**.** *In this case,* τ_{x_0} *and* τ_{x_1} *generate a symmetric group* S_3 *and* $\langle x_0, x_1\rangle = \frac{13}{2^{10}}$*. Let* $x_2 := \tau_{x_0}(x_1)$ *and* $u := \frac{2^6}{3^3\cdot 5}(2x_0+2x_1+x_2-2^4x_0\cdot x_1)$*. Then, u is a conformal vector of central charge* $\frac{4}{5}$*,* $\mathcal{G} = \mathrm{Span}_{\mathbb{R}}\{x_0, x_1, x_2, u\}$ *and* $\dim \mathcal{G} = 4$*. For* $\{i, j, k\} = \{0, 1, 2\}$*, the multiplication and the bilinear form are given by*

$$x_i \cdot x_j = \frac{1}{2^4}(2x_i + 2x_j + x_k) - \frac{135}{2^{10}}u, \tag{4.0.3}$$

$$x_i \cdot u = \frac{2}{3^2}(2x_i - x_j - x_k) + \frac{5}{2^4}u, \tag{4.0.4}$$

$$u \cdot u = 2u, \tag{4.0.5}$$

and

$$\langle x_i, x_j\rangle = \frac{13}{2^{10}}, \quad \langle x_i, u\rangle = \frac{1}{2^4}, \quad \langle u, u\rangle = \frac{2}{5}. \tag{4.0.6}$$

Moreover, we have

$$\tau_{x_i}(x_j) = x_k, \quad \text{and} \quad \tau_{x_i}(u) = u. \tag{4.0.7}$$

For $i \in \mathbb{Z}_3$*, the fixed point subalgebra* $\mathcal{G}^{\tau_{x_i}}$ *has dimension 3 and is spanned by* x_i*,* $x_j + x_k$ *and u. Moreover we have*

$$\sigma_{x_i}(x_j + x_k) = -\frac{3x_i}{2^4} + \frac{x_j + x_k}{2^2} + \frac{135u}{2^7}, \tag{4.0.8}$$

$$\sigma_{x_i}(u) = \frac{2x_i}{3^2} + \frac{8(x_j + x_k)}{3^2} - \frac{u}{2^2}. \tag{4.0.9}$$

$I = I_0 = I_1 = \{x_0, x_1, x_2\}$ *is the set of all Ising vectors in* $\mathcal{G}$*.*

We call the ordered set (x_0, x_1, x_2, u) *a* normal $\mathcal{G}U_{3A}$ basis.

4.0.8. $\mathcal{G}U_{3C}$. *In* $\mathcal{G}U_{3C}$, τ_{x_0} *and* τ_{x_1} *generate a symmetric group* S_3 *and* $\langle x_0, x_1\rangle = \frac{1}{2^8}$. *Let* $x_2 := \tau_{x_0}(x_1)$. *Then,* $\mathcal{G} = \mathrm{Span}_{\mathbb{R}}\{x_0, x_1, x_2\}$ *and* $\dim \mathcal{G} = 3$. *The multiplication and the bilinear form are given by*

$$x_i \cdot x_j = \frac{1}{2^5}(x_i + x_j - x_k), \quad \text{and} \quad \langle x_i, x_j\rangle = \frac{1}{2^8}, \tag{4.0.10}$$

where $\{i, j, k\} = \{0, 1, 2\}$. *In this case, we also have* $\tau_{x_i}(x_j) = x_k$ *and* $I = I_0 = I_1 = \{x_0, x_1, x_2\}$ *is the set of all Ising vectors in* $\mathcal{G}$.

The fixed point subalgebra $\mathcal{G}^{\tau_{x_i}}$ *has dimension 2 and is spanned by* x_i *and* $x_j + x_k$. *Moreover we have* $\sigma_{x_i}(x_j + x_k) = x_j + x_k$. *We call* (x_0, x_1, x_2) *a* normal $\mathcal{G}U_{3C}$ basis.

4.0.9. $\mathcal{G}U_{4A}$. *In* $\mathcal{G}U_{4A}$, τ_{x_0} *and* τ_{x_1} *generate a Klein's 4-group and* $\langle x_0, x_1\rangle = \frac{1}{2^7}$. *Let* $x_2 := \tau_{x_1}(x_0)$, $x_3 := \tau_{x_0}(x_1)$ *and* $\mu := x_0 + x_1 + \frac{1}{3}x_2 + \frac{1}{3}x_3 - \frac{2^5}{3}x_0 \cdot x_1$. *Then* μ *is a conformal vector of central charge 1 and* $\mathcal{G} = \mathrm{Span}_{\mathbb{R}}\{x_0, x_1, x_2, x_3, \mu\}$. *The dimension of* $\mathcal{G}$ *is 5 and the multiplication and the bilinear form are given as the following.*

For $k \equiv i + 2 \pmod 4$, *the pair* (x_i, x_k) *forms a normal* $\mathcal{G}U_{2B}$ *basis. The product structure and the bilinear form between* x_i *and* x_k *are then shown as in* $\mathcal{G}U_{2B}$.

For $j \equiv i + 1 \pmod 4$, $\{i, j, k, l\} = \{0, 1, 2, 3\}$, *we have*

$$x_i \cdot x_j = \frac{1}{2^5}(3x_i + 3x_j + x_k + x_l - 3\mu) \quad \text{and} \quad \langle x_i, x_j\rangle = \frac{1}{2^7}. \tag{4.0.11}$$

For $k \equiv i + 2 \pmod 4$, $\{i, j, k, l\} = \{0, 1, 2, 3\}$, *we have*

$$x_i \cdot \mu = \frac{1}{2^3}(5x_i - 2x_j - x_k - 2x_l + 3\mu) \quad \text{and} \quad \langle x_i, \mu\rangle = \frac{3}{2^5}. \tag{4.0.12}$$

We also have

$$\mu \cdot \mu = 2\mu \quad \text{and} \quad \langle \mu, \mu\rangle = \frac{1}{2}. \tag{4.0.13}$$

Moreover, $\tau_{x_i}(x_j) = x_l,\ \tau_{x_i}(x_l) = x_j$ *for* $j \equiv i+1 \pmod 4$, $l \equiv i-1 \pmod 4$ *and* $\tau_{x_i}(\mu) = \mu$ *for* $i \in \{0,1,2,3\}$. *In this case,* $I_0 = \{x_0, x_2\}$, $I_1 = \{x_1, x_3\}$, *and there are exactly* 4 *Ising vectors in* $\mathcal{G}$.

The fixed point subalgebra $\mathcal{G}^{\tau_{x_i}}$ *has dimension 4 and is spanned by* x_i, x_k, μ *and* $x_j + x_l$, *where* $k \equiv i+2 \pmod 4$, $j \equiv i+1 \pmod 4$ *and* $l \equiv i-1 \pmod 4$. *In addition,*

$$\sigma_{x_i}(x_j + x_l) = -\frac{x_i}{2^2} - \frac{x_k}{2^2} + \frac{x_j + x_l}{2} + \frac{3\mu}{2^2},$$

$$\sigma_{x_i}(\mu) = \frac{x_i}{2} + \frac{x_k}{2} + (x_j + x_l) - \frac{\mu}{2}.$$

We call the ordered set $(x_0, x_1, x_2, x_3, \mu)$ *a* normal $\mathcal{G}U_{4A}$ basis.

4.0.10. $\mathcal{G}U_{4B}$**.** *In* $\mathcal{G}U_{4B}$, τ_{x_0} *and* τ_{x_1} *generate a Klein's 4-group and* $\langle x_0, x_1\rangle = \frac{1}{2^8}$. *Let* $x_2 := \tau_{x_1}(x_0)$, $x_3 := \tau_{x_0}(x_1)$, *and* $x := -x_0 - x_1 + x_2 + x_3 + 2^5 x_0 \cdot x_1$. *Then* x *is an Ising vector and* $\mathcal{G} = \mathrm{Span}_{\mathbb{R}}\{x_0, x_1, x_2, x_3, x\}$. *The dimension of* $\mathcal{G}$ *is 5.*

The multiplication and the bilinear form are given as the following.

For $k \equiv i + 2 \pmod 4$, *the triple* (x_i, x_k, x) *forms a normal* $\mathcal{G}U_{2A}$ *basis for* $\mathcal{G}\{x_i, x_k\}$, *and hence the product structure and the bilinear form between* x_i, x_k, x *are shown as in* $\mathcal{G}U_{2A}$.

For $j \equiv i + 1 \pmod 4$, $\{i, j, k, l\} = \{0, 1, 2, 3\}$, *we have*

$$x_i \cdot x_j = \frac{1}{2^5}(x_i + x_j - x_k - x_l + x), \qquad \textit{and} \qquad \langle x_i, x_j\rangle = \frac{1}{2^8}.$$

Moreover, we have $\tau_{x_i}(x_j) = x_l,\ \tau_{x_i}(x_l) = x_j$ *for* $j \equiv i+1 \pmod 4$, $l \equiv i-1 \pmod 4$ *and* $\tau_{x_i}(x) = x$ *for* $i \in \{0,1,2,3\}$. *In this case,* $I_0 = \{x_0, x_2\}$, $I_1 = \{x_1, x_3\}$ *and there are 5 Ising vectors in* $\mathcal{G}$.

The fixed point subalgebra $\mathcal{G}^{\tau_{x_i}}$ *has dimension 4 and is spanned by* x_i, x_k, x *and* $x_j + x_l$, *where* $k \equiv i + 2 \pmod 4$, $\{i, j, k, l\} = \{0, 1, 2, 3\}$. *Moreover we have*

$$\sigma_{x_i}(x_j + x_l) = (x_j + x_l) + \frac{x_k}{2^2} - \frac{x}{2^2}.$$

We call the ordered set (x_0, x_1, x_2, x_3, x) *a* normal $\mathcal{G}U_{4B}$ basis.

4.0.11. $\mathcal{G}U_{5A}$**.** *In* $\mathcal{G}U_{5A}$, τ_{x_0} *and* τ_{x_1} *generate a dihedral group of order 10. Let* $x_2 := \tau_{x_1}(x_0)$, $x_3 := \tau_{x_2}(x_1)$, $x_4 := \tau_{x_0}(x_1)$, *and* $\nu := -x_0 - x_1 + x_2 + x_3 + 2^5 x_0 \cdot x_1$. *Then* $\mathcal{G} = \mathrm{Span}_{\mathbb{R}}\{x_0, x_1, x_2, x_3, x_4, \nu\}$ *and* $\dim \mathcal{G} = 6$. *In this case, the vector* ν *is fixed by* τ_{x_i} *for all* $i \in \{0, 1, 2, 3, 4\}$. *The multiplication and the bilinear form are given as the following.*

For $j \equiv i \pm 1 \pmod 5$, $\{i, j, k, l, m\} = \{0, 1, 2, 3, 4\}$,

$$x_i \cdot x_j = \frac{1}{2^6}(3x_i + 3x_j - x_k - x_l - x_m) + 2\nu.$$

For $k \equiv i \pm 2 \pmod 5$, $\{i, j, k, l, m\} = \{0, 1, 2, 3, 4\}$,

$$x_i \cdot x_k = \frac{1}{2^6}(3x_i - x_j + 3x_k - x_l - x_m) - 2\nu.$$

For $j \equiv i + 1 \pmod 5$, $m \equiv i - 1 \pmod 5$, $k \equiv i + 2 \pmod 5$, $l \equiv i - 2 \pmod 5$,

$$x_i \cdot \nu = \frac{7}{2^{11}}(x_j - x_k - x_l + x_m) + \frac{7}{2^4}\nu.$$

We also have

$$\nu \cdot \nu = \frac{5^2 \cdot 7}{2^{18}}(x_0 + x_1 + x_2 + x_3 + x_4).$$

For $j \neq i$, *we have*

$$\langle x_i, x_j \rangle = \frac{3}{2^9}, \quad \langle x_i, \nu \rangle = 0, \quad \langle \nu, \nu \rangle = \frac{5^3 \cdot 7}{2^{21}}.$$

Moreover, $\tau_{x_i}(x_j) = x_m$, *for* $j + m \equiv 2i \pmod 5$ *and* $\tau_{x_i}(\nu) = \nu$ *for* $i \in \{0, 1, 2, 3, 4\}$. *In this case,* $I = I_0 = I_1 = \{x_0, \ldots, x_4\}$ *and there are exactly* 5 *Ising vectors in* $\mathcal{G}$.

The fixed point subalgebra $\mathcal{G}^{\tau_{x_i}}$ *has dimension 4 and is spanned by* x_i, ν, $x_j + x_m$ *and* $x_k + x_l$ *where* $j \equiv i + 1 \pmod 5$, $m \equiv i - 1 \pmod 5$, $k \equiv i + 2 \pmod 5$ *and* $l \equiv i - 2 \pmod 5$. *Moreover we have*

$$\sigma_{x_i}(x_j + x_m) = \frac{x_j + x_m}{2^3} + \frac{7(x_k + x_l)}{2^3} - 16\nu,$$

$$\sigma_{x_i}(x_k + x_l) = \frac{7(x_j + x_m)}{2^3} + \frac{x_k + x_l}{2^3} + 16\nu,$$

$$\sigma_{x_i}(\nu) = \frac{-7(x_j + x_m)}{2^9} + \frac{7(x_k + x_l)}{2^9} - \frac{3}{4}\nu.$$

We call the ordered set $(x_0, x_1, x_2, x_3, x_4, \nu)$ *a* normal $\mathcal{G}U_{5A}$ basis.

4.0.12. $\mathcal{G}U_{6A}$. *In* $\mathcal{G}U_{6A}$, $(\tau_{x_0}\tau_{x_1})^3 = 1$, *and* $\tau_{x_0}\tau_{x_1}(x_0) \neq x_1$. *Let* $x_2 := \tau_{x_1}(x_0)$, $x_3 := \tau_{x_2}(x_1)$, $x_4 := \tau_{x_3}(x_2)$, $x_5 := \tau_{x_0}(x_1)$, $x := x_0 + x_2 - 2^2 x_0 \cdot x_2$ *and* $\mu := \frac{2^6}{3^3 \cdot 5}(2x_0 + 2x_2 + x_4 - 2^4 x_0 \cdot x_2)$. *Then* x *is an Ising vector and* u *is a conformal vector of central charge* $\frac{4}{5}$. *Moreover, we have* $\mathcal{G} = \mathrm{Span}_{\mathbb{R}}\{x_0, x_1, x_2, x_3, x_4, x_5, x, \mu\}$ *and* $\dim \mathcal{G} = 8$. *The multiplication and the bilinear form are given as the following.*

- *For* $k \equiv i + 2 \,(\mathrm{mod}\, 6)$, $m \equiv i - 2 \,(\mathrm{mod}\, 6)$, *the quadruple* (x_i, x_k, x_m, μ) *forms a normal* $\mathcal{G}U_{3A}$ *basis. Hence their structures are shown as in* $\mathcal{G}U_{3A}$.
- *For* $l \equiv i + 3 \,(\mathrm{mod}\, 6)$, *the quadruple* (x_i, x_l, x) *forms a normal* $\mathcal{G}U_{2A}$ *basis. In particular, we have* $x_i \cdot x_l = \frac{1}{4}(x_i + x_l - x)$.
- *For* $j \equiv i + 1 \,(\mathrm{mod}\, 6)$, $\{i, j, k, l, m, n\} = \{0, 1, 2, 3, 4, 5\}$, *we have*

$$x_i \cdot x_j = \frac{1}{2^5}(x_i + x_j - x_k - x_l - x_m - x_n + x) + \frac{45}{2^{10}}\mu. \tag{4.0.14}$$

We also have

$$x \cdot \mu = 0, \qquad \langle x, \mu \rangle = 0, \tag{4.0.15}$$

and

$$\langle x_i, x_j \rangle = \frac{5}{2^{10}} \qquad \text{for } j \equiv i + 1 \,(\mathrm{mod}\, 6). \tag{4.0.16}$$

Moreover, for $i, j \in \mathbb{Z}_6$, *we have*

$$\tau_{x_i}(x_j) = x_{2i-j}. \tag{4.0.17}$$

The fixed point subalgebra $\mathcal{G}^{\tau_{x_i}}$ *has dimension 6 and is spanned by* x_i, x_l, x, μ, $x_j + x_n$, $x_k + x_m$, *where* $l \equiv i+3 \pmod 6$, $j \equiv i+1 \pmod 6$, $n \equiv i-1 \pmod 6$, $k \equiv i+2 \pmod 6$, $m \equiv i-2 \pmod 6$. *Moreover we have*

$$\sigma_{x_i}(x_j + x_n) = \frac{x_i}{2^4} + \frac{x_l}{2^2} + (x_j + x_n) + \frac{x_k + x_m}{2^2} - \frac{x}{2^2} - \frac{45\mu}{2^7}.$$

We call the ordered set $(x_0, x_1, x_2, x_3, x_4, x_5, x, \mu)$ *a* normal $\mathcal{G}U_{6A}$ basis.

REMARK 4.5. By Sakuma's Theorem (Theorem 4.4), it is easy to see that $a = b$ if and only if $\langle a, b\rangle = \frac{1}{2^2}$ for any 2 Ising vectors a, b.

CHAPTER 5

Griess algebras generated by 3 Ising vectors of central $2A$-type

Since Griess algebras generated by 2 Ising vectors are classified. We may want to classify Griess algebras generated by 3 Ising vectors. However, a group generated by 3 involutions can be very large and the situation could be very complicated. Here we concentrate on Griess algebras generated by 3 Ising vectors of central $2A$-type (Definition 5.2). In this case, we can classify all possibilities and each has the corresponding VOA constructed in [**HLY1**].

NOTATION 5.1. *Let S be a subset of $\mathcal{G} = V_2$, we use $\mathcal{G}S$ to denote the (Griess) subalgebra generated by S. For example, $\mathcal{G}\{x, y\}$ denotes the (Griess) subalgebra generated by x and y.*

DEFINITION 5.2. *Let V be a VOA satisfying Assumption 1 and let e, x_0, x_1 be Ising vectors in V_2. The set $\{e, x_0, x_1\}$ is said to be of* central $2A$-type *if $\mathcal{G}\{e, x_0\} \cong \mathcal{G}\{e, x_1\} \cong \mathcal{G}U_{2A}$. In this case, τ_e commutes with τ_{x_0} and τ_{x_1}.*

The following lemma can be found in [**HLY1**], which is proved by Matsuo [**Ma**].

LEMMA 5.3. *Suppose that V is a VOA satisfying Assumption 1. Let x_0, x_1, x_2 and e be Ising vectors of V such that (x_0, x_1, x_2) forms a normal $\mathcal{G}U_{2A}$ basis (recall 4.0.5). Then it is impossible that $\mathcal{G}\{e, x_i\} \cong \mathcal{G}U_{2A}$ for all $i = 0, 1, 2$.*

In [**HLY1**], certain VOA generated by 3 Ising vectors of central $2A$-type are constructed. There are 5 cases and they are denoted by $V_{\mathbb{B}(1A)}$, $V_{\mathbb{B}(2B)}$, $V_{\mathbb{B}(3A)}$, $V_{\mathbb{B}(4B)}$, and $V_{\mathbb{B}(2C)}$. We denote their Griess algebras by $\mathcal{G}V_{\mathbb{B}(nX)}$.

The next theorem is the main theorem of Chapter 5, which shows that there are only five possible structures for Griess algebra generated by 3 Ising vectors of central $2A$-type.

5.1. Main theorem

THEOREM 5.4. *Let V be a VOA satisfying Assumption 1 and let e, x_0, x_1 be Ising vectors of central $2A$-type. Then the Griess subalgebra $\mathcal{G}$ generated by e, x_0, x_1 is isomorphic to one of the following algebras.*

(1) $\mathcal{G}V_{\mathbb{B}(1A)}$. *In this case, $\mathcal{G}\{e, x_0\} \cong \mathcal{G}\{e, x_1\}$. Then $\mathcal{G}$ is generated by e and x_0. By our assumption, $\mathcal{G}$ is isomorphic to $\mathcal{G}U_{2A}$ in the previous chapter and* $\dim \mathcal{G} = 3$.

(2) $\mathcal{G}V_{\mathbb{B}(2B)}$. *The algebra $\mathcal{G}V_{\mathbb{B}(2B)}$ is isomorphic to the Griess algebra of $V^+_{\sqrt{2}A_2}$. In this case, $\mathcal{G} = \mathcal{G}V_{\mathbb{B}(2B)} = \mathrm{Span}_{\mathbb{R}}\{e, f, f', g, g', h\}$ and $\dim \mathcal{G} = 6$, where (e, f, f'), (e, g, g'), (h, f, g), and (h, f', g') form normal $\mathcal{G}U_{2A}$ bases of $\mathcal{G}\{e, x_0\}$, $\mathcal{G}\{e, x_1\}$, $\mathcal{G}\{f, g\}$, and $\mathcal{G}\{f', g'\}$ respectively. In addition, (e, h), (f, g'), and (f', g) form normal $\mathcal{G}U_{2B}$ basis for $\mathcal{G}\{e, h\}$, $\mathcal{G}\{f, g'\}$, and $\mathcal{G}\{f', g\}$ respectively. The multiplication and the bilinear form can be obtained via the structures of $\mathcal{G}U_{2A}$ and $\mathcal{G}U_{2B}$ (c.f. Figure 1).*

(3) $\mathcal{G}V_{\mathbb{B}(2C)}$. *In this case, x_0 and x_1 generate $\mathcal{G}U_{4B}$ and $\mathcal{G} = \mathcal{G}U_{4B}$ in the previous chapter with $e = x$. The dimension of $\mathcal{G}$ is 5 (c.f. Figure 3).*

(4) $\mathcal{G}V_{\mathbb{B}(3A)}$. *In this case, x_0 and x_1 generate $\mathcal{G}U_{6A}$ or $\mathcal{G}U_{3A}$ and $\mathcal{G}$ is isomorphic to $\mathcal{G}U_{6A}$ as described in the previous chapter with $e = x$ and* $\dim \mathcal{G} = 8$ *(c.f. Figure 2).*

(5) $\mathcal{G}V_{\mathbb{B}(4B)}$. *In this case, x_0 and x_1 generate $\mathcal{G}U_{4A}$. Let $(x_0, x_1, x_2, x_3, \mu)$ be a normal $\mathcal{G}U_{4A}$ basis and $y_i := \tau_e(x_i)$. Then $y_2 \in \mathcal{G}^{\tau_{x_0}}$ and let $e' := \sigma_{x_0}(y_2)$. The subalgebra $\mathcal{G}\{x_0, y_1\}$ is also isomorphic to $\mathcal{G}U_{4A}$. Let $\mu' \in \mathcal{G}\{x_0, y_1\}$ such that*

$(x_0, y_1, x_2, y_3, \mu')$ *forms a normal* $\mathcal{G}U_{4A}$ *basis. Then,*

$$\mathcal{G} = \mathrm{Span}_{\mathbb{R}}\{e, e', x_0, x_1, x_2, x_3, y_0, y_1, y_2, y_3, \mu, \mu'\}$$

with an extra relation $x_0+x_1+x_2+x_3+y_0+y_1+y_2+y_3-e-e'-\frac{3}{2}\mu-\frac{3}{2}\mu' = 0$. *The dimension of* $\mathcal{G}$ *is 11. Elements* $e, e', x_0, x_1, x_2, x_3, y_0, y_1, y_2, y_3$ *are Ising vectors and* μ, μ' *are conformal vectors with central charge 1.*

The structures can be summarized as follows (c.f. Figure 4).

- *The ordered sets* $(x_0, x_1, x_2, x_3, \mu)$, $(y_0, y_1, y_2, y_3, \mu)$, $(x_0, y_1, x_2, y_3, \mu')$, *and* $(y_0, x_1, y_2, x_3, \mu')$ *form normal* $\mathcal{G}U_{4A}$ *bases.*
- *The triples* (e, x_i, y_i) *and* (e', x_i, y_j) *form normal* $\mathcal{G}U_{2A}$ *bases for* $i \in \{0, 1, 2, 3\}$, $j \equiv i+2 \pmod 4$.
- *The pair* (e, e') *forms a normal* $\mathcal{G}U_{2B}$ *basis.*
- *The remaining structures are listed below.*

  $$\mu \cdot e = 0,\ \mu \cdot e' = 0,\ \mu' \cdot e = 0,\ \mu' \cdot e' = 0,\ \mu \cdot \mu' = 0,$$

  and

  $$\langle \mu, e\rangle = 0,\ \langle \mu, e'\rangle = 0,\ \langle \mu', e\rangle = 0,\ \langle \mu', e'\rangle = 0,\ \langle \mu, \mu'\rangle = 0.$$

  In addition, we have

  $$\tau_e(\mu) = \mu,\ \tau_e(\mu') = \mu',\ \tau_{e'}(\mu) = \mu,\ \tau_{e'}(\mu') = \mu',$$

  and

  $$\sigma_e(\mu) = \mu,\ \sigma_e(\mu') = \mu',\ \sigma_{e'}(\mu) = \mu,\ \sigma_{e'}(\mu') = \mu'.$$

5.2. Proof of the main theorem

In the following, we will give a proof for Theorem 5.4.

NOTATION 5.5. *Let* $x_0' := \sigma_e(x_0)$ *and* $x_1' := \sigma_e(x_1)$. *Then* (e, x_0, x_0') *and* (e, x_1, x_1') *form normal* $\mathcal{G}U_{2A}$ *bases for* $\mathcal{G}\{e, x_0\}$ *and* $\mathcal{G}\{e, x_1\}$.

By Theorem 4.4, there are 9 possibilities for $\mathcal{G}\{x_0, x_1\}$. We will analyze each case in details.

5.2.1. Case 1. $\mathcal{G}\{x_0, x_1\} \cong \mathcal{G}U_{1A}$**.** In this case, $x_0 = x_1$. Then $\mathcal{G}\{e, x_0, x_1\} = \mathcal{G}\{e, x_0\} = \mathcal{G}U_{2A}$. This algebra is isomorphic to $\mathcal{G}V_{\mathbb{B}(1A)}$.

5.2.2. Case 2. $\mathcal{G}\{x_0, x_1\} \cong \mathcal{G}U_{2A}$**.**

LEMMA 5.6. *Let* $h := \sigma_{x_0}(x_1) = x_0 + x_1 - 2^2 x_0 \cdot x_1$ *(by (4.0.1)). Then* $\langle e, h\rangle = 0$ *or* $\frac{1}{2^2}$.

PROOF. Since τ_e is trivial on $\mathcal{G}\{x_0, x_1, e\} = \mathcal{G}$, we have $\tau_e(h) = h$ and $\mathrm{Gp}\langle \tau_e, \tau_h\rangle \cdot \{h\} = \{h\}$. Hence $\mathcal{G}\{e, h\}$ is isomorphic to $\mathcal{G}U_{1A}$, $\mathcal{G}U_{2A}$ or $\mathcal{G}U_{2B}$ by Theorem 4.4.

Since (x_0, x_1, h) forms a normal $\mathcal{G}U_{2A}$ basis for $\mathcal{G}\{x_0, x_1\}$ and $\mathcal{G}\{e, x_0\} \cong \mathcal{G}\{e, x_1\} \cong \mathcal{G}U_{2A}$ by our assumption, $\mathcal{G}\{e, h\}$ cannot be isomorphic to $\mathcal{G}U_{2A}$ by Lemma 5.3. Hence we have $\mathcal{G}\{e, h\} \cong \mathcal{G}U_{2B}$ or $\mathcal{G}U_{1A}$, i.e. $\langle e, h\rangle = 0$ or $\frac{1}{2^2}$. □

LEMMA 5.7. *We have* $\langle x_0, x_1'\rangle = \langle e, h\rangle$. *Hence* $\langle x_0, x_1'\rangle = 0$ *or* $\frac{1}{2^2}$.

PROOF. Since $x_1' = \sigma_e(x_1) = e + x_1 - 2^2 e \cdot x_1$ (by (4.0.1)), we have

$$\begin{aligned}
\langle x_0, x_1'\rangle &= \langle x_0, e + x_1 - 2^2 e \cdot x_1\rangle = \langle x_0, e\rangle + \langle x_0, x_1\rangle - 2^2\langle x_0, e \cdot x_1\rangle \\
&= \frac{1}{2^5} + \frac{1}{2^5} - 2^2\langle e, x_0 \cdot x_1\rangle = \frac{1}{2^5} + \frac{1}{2^5} - \langle e, x_0 + x_1 - h\rangle \\
&= \frac{1}{2^5} + \frac{1}{2^5} - \frac{1}{2^5} - \frac{1}{2^5} + \langle e, h\rangle = \langle e, h\rangle
\end{aligned}$$

as desired. □

PROPOSITION 5.8. (1) *If* $\langle e, h\rangle = \frac{1}{2^2}$, *then* $\mathcal{G}\{e, x_0, x_1\} \cong \mathcal{G}U_{2A}$.

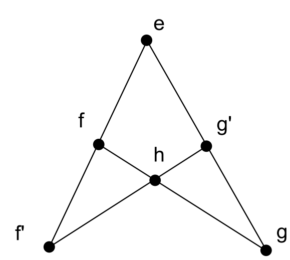
(2) *If* $\langle e, h\rangle = 0$, *then* $\dim \mathcal{G}\{e, x_0, x_1\} = 6$ *and* $\mathcal{G}\{e, x_0, x_1\} \cong \mathcal{G}V_{\mathbb{B}(2B)}$.

PROOF. (1) When $\langle e, h\rangle = \langle x_0, x_1'\rangle = \frac{1}{2^2}$, we have $e = h$ and $x_0 = x_1'$ by Remark 2.25. Therefore $\mathcal{G}\{e, x_0, x_1\} = \mathcal{G}\{e, x_1', x_1\} \cong \mathcal{G}U_{2A}$.

(2) When $\langle e, h\rangle = \langle x_0, x_1'\rangle = 0$, we have $\mathcal{G}\{e, h\} \cong \mathcal{G}\{x_0, x_1'\} \cong \mathcal{G}U_{2B}$. Set $f := x_0$, $g := x_1$, $f' := x_0'$, $g' := x_1'$. We have a normal $\mathcal{G}U_{2A}$ basis (f, g, h) for $\mathcal{G}\{x_0, x_1\} = \mathcal{G}\{f, g\}$ and normal $\mathcal{G}U_{2B}$ bases (e, h) and (f, g') for $\mathcal{G}\{e, h\}$ and $\mathcal{G}\{f, g'\}$ respectively. Since τ_e is trivial on $\mathcal{G}$, we can apply σ_e to the normal $\mathcal{G}U_{2A}$ basis (f, g, h) to get another normal $\mathcal{G}U_{2A}$ basis (f', g', h) and apply σ_e to the normal $\mathcal{G}U_{2B}$ basis (f, g') to get a normal $\mathcal{G}U_{2B}$ basis (f', g). Therefore, there are 4 normal $\mathcal{G}U_{2A}$ bases and 3 normal $\mathcal{G}U_{2B}$ bases. The structure is summarized in Figure 1. In Figure 1, any three collinear points form a normal $\mathcal{G}U_{2A}$ basis and any 2 points not joined by a line form a normal $\mathcal{G}U_{2B}$ basis.

Hence $\mathcal{G} = \mathrm{Span}_{\mathbb{R}}\{e, h, f, f', g, g'\}$ is closed under multiplication. This algebra is isomorphic to $\mathcal{G}V_{\mathbb{B}(2B)}$.

FIGURE 1. Configuration for $\mathcal{G}V_{\mathbb{B}(2B)}$



To prove that $\{e, h, f, f', g, g'\}$ is linearly independent, we can compute $\det(\langle a_i, a_j\rangle)$ for $a_i \in \{e, h, f, f', g, g'\}$. By computer, we verify that $\det(\langle a_i, a_j\rangle) = \frac{3^3}{2^{17}} \neq 0$. Hence $\mathcal{G}V_{\mathbb{B}(2B)}$ have dimension 6.

□

5.2.3. Case 3. $\mathcal{G}\{x_0, x_1\} \cong \mathcal{G}U_{2B}$. In this case, we have

$$\begin{aligned}\langle x_0, x_1' \rangle &= \langle x_0, e + x_1 - 2^2 e \cdot x_1 \rangle = \langle x_0, e \rangle + \langle x_0, x_1 \rangle - 2^2 \langle x_0, e \cdot x_1 \rangle \\ &= \frac{1}{2^5} + 0 - 2^2 \langle e, x_0 \cdot x_1 \rangle = \frac{1}{2^5}.\end{aligned}$$

That is to say, $\mathcal{G}\{x_0, x_1'\}$ is isomorphic to $\mathcal{G}U_{2A}$. Moreover, it is easy to see that $\mathcal{G}\{e, x_0, x_1\}$ $= \mathcal{G}\{e, x_0, x_1'\}$ since $\mathcal{G}\{e, x_1\} = \mathcal{G}\{e, x_1'\} \cong \mathcal{G}U_{2A}$. Hence, by Case 2, we have $\mathcal{G}\{e, x_0, x_1\} = \mathcal{G}\{e, x_0, x_1'\} \cong \mathcal{G}V_{\mathbb{B}(2B)}$.

5.2.4. Case 4. $\mathcal{G}\{x_0, x_1\} \cong \mathcal{G}U_{3A}$. Let $x_2 := \tau_{x_0}(x_1)$ and $u := \frac{2^6}{3^3 \cdot 5}(2x_0 + 2x_1 + x_2 - 2^4 x_0 \cdot x_1)$. Then (x_0, x_1, x_2, u) forms a normal $\mathcal{G}U_{3A}$ basis and

$$\langle e, x_2 \rangle = \langle \tau_{x_1}(e), \tau_{x_1}(x_2) \rangle = \langle e, x_0 \rangle = \frac{1}{2^5}.$$

Hence $\mathcal{G}\{e, x_2\} \cong \mathcal{G}U_{2A}$. Let $x_2' := \sigma_e(x_2)$. Then (e, x_2, x_2') forms a normal $\mathcal{G}U_{2A}$ basis.

LEMMA 5.9. *We have $\tau_{x_i}\sigma_e = \sigma_e\tau_{x_i}$ and $\tau_{x_i} = \tau_{x_i'}$ for any $i = 0, 1$.*

PROOF. Since τ_{x_1} fixes e, we have $\tau_{x_1}\sigma_e\tau_{x_1} = \sigma_{\tau_{x_1}(e)} = \sigma_e$, which implies $\tau_{x_1}\sigma_e = \sigma_e\tau_{x_1}$. Therefore, $\tau_{x_1} = \sigma_e\tau_{x_1}\sigma_e = \tau_{\sigma_e(x_1)} = \tau_{x_1'}$. Similarly we also have $\tau_{x_0}\sigma_e = \sigma_e\tau_{x_0}$ and $\tau_{x_0} = \tau_{x_0'}$. □

LEMMA 5.10. *The set $\mathrm{Gp}\langle \tau_{x_0}, \tau_{x_1'} \rangle \cdot \{x_0, x_1'\} = \{x_0, x_1, x_2, x_0', x_1', x_2'\}$.*

Proof. By Lemma 5.9, $\tau_{x_1'}(x_0) = \tau_{x_1}(x_0) = x_2$, $\tau_{x_0}(x_2) = x_1$, and $\mathrm{Gp}\langle \tau_{x_0}, \tau_{x_1'} \rangle \cdot \{x_0\} = \{x_0, x_1, x_2\}$. Moreover,

$$
\begin{aligned}
\mathrm{Gp}\langle \tau_{x_0}, \tau_{x_1'} \rangle \cdot \{x_1'\} &= \mathrm{Gp}\langle \tau_{x_0}, \tau_{x_1'} \rangle \cdot \{\sigma_e(x_1)\} \\
&= \sigma_e \cdot (\mathrm{Gp}\langle \tau_{x_0}, \tau_{x_1'} \rangle \cdot \{x_1\}) \\
&= \sigma_e \cdot \{x_0, x_1, x_2\} \\
&= \{x_0', x_1', x_2'\}
\end{aligned} \tag{5.2.1}
$$

and thus we have the desired result. □

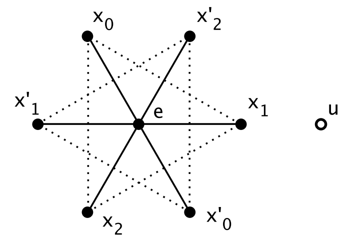
Lemma 5.11. *For any $i, j \in \{0, 1, 2\}$, $x_i' \neq x_j$.*

Proof. Clearly, $x_i \neq x_i'$ since $\mathcal{G}\{x_i, x_i'\} \cong \mathcal{G}U_{2A}$. Suppose $x_i' = x_j$ for some $i \neq j$. Then $\mathcal{G}\{x_i, x_i'\} = \mathcal{G}\{x_i, x_j\} \cong \mathcal{G}U_{3A}$ by our assumption. It is absurd since $\mathcal{G}\{x_i, x_i'\} \cong \mathcal{G}U_{2A}$. □

Proposition 5.12. *We have $\mathcal{G}\{x_0, x_1, e\} \cong \mathcal{G}U_{6A} \cong \mathcal{G}V_{\mathbb{B}(3A)}$.*

Proof. By Lemma 5.10 and 5.11, there are at least 6 distinct Ising vectors in $\mathcal{G}\{x_0, x_1'\}$ and hence $\mathcal{G}\{x_0, x_1'\} \cong \mathcal{G}U_{6A}$ by Theorem 4.4. By Lemma 5.10, we have $\{x_0, x_1, x_2, x_0', x_1', x_2'\} \subset \mathcal{G}\{x_0, x_1'\}$ and thus $e \in \mathcal{G}\{x_1, x_1'\} \subset \mathcal{G}\{x_0, x_1'\}$. Hence the Griess algebra $\mathcal{G}\{x_0, x_1'\}$ contains $\mathcal{G}\{x_0, x_1, e\}$ and $\mathcal{G}\{x_0, x_1, e\} = \mathcal{G}\{x_0, x_1'\} = \mathrm{Span}_{\mathbb{R}}\{x_0, x_1', x_2, x_0', x_1, x_2', e, u\} \cong \mathcal{G}U_{6A}$. This algebra is isomorphic to $\mathcal{G}V_{\mathbb{B}(3A)}$. The structure is shown in Figure 2, where three collinear points joined by a solid line form a normal $\mathcal{G}U_{2A}$ basis and the vertices of a dotted triangle form a normal $\mathcal{G}U_{3A}$ basis with u. □

FIGURE 2. Configuration for $\mathcal{G}V_{\mathbb{B}(3A)}$



5.2.5. Case 5. $\mathcal{G}\{x_0, x_1\} \cong \mathcal{G}U_{3C}$. Let $x_2 := \tau_{x_1}(x_0)$, $x_2' := \sigma_e(x_2) = e + x_2 - 2^2 e \cdot x_2$. As in (5.9), we also have $\tau_{x_i} = \tau_{x_i'}$ for $i = 0, 1, 2$. Then,

$$\begin{aligned} \langle x_0, x_1' \rangle &= \langle x_0, e + x_1 - 2^2 e \cdot x_1 \rangle \\ &= \langle x_0, e \rangle + \langle x_0, x_1 \rangle - 2^2 \langle e, x_0 \cdot x_1 \rangle \\ &= \frac{1}{2^5} + \frac{1}{2^8} - 2^2 \left\langle e, \frac{1}{2^5}(x_0 + x_1 - x_2) \right\rangle \\ &= \frac{1}{2^5} + \frac{1}{2^8} - 2^2 \cdot \frac{1}{2^5} \left(\frac{1}{2^5} + \frac{1}{2^5} - \frac{1}{2^5} \right) \\ &= \frac{1}{2^5}. \end{aligned}$$

It implies $\mathcal{G}\{x_0, x_1'\}$ is isomorphic to $\mathcal{G}U_{2A}$, and we have $\tau_{x_1'}(x_0) = x_0$. On the other hand, $\tau_{x_1'}(x_0) = \tau_{x_1}(x_0) = x_2 \neq x_0$. That is a contradiction. Hence there is no such Griess algebra.

5.2.6. Case 6. $\mathcal{G}\{x_0, x_1\} \cong \mathcal{G}U_{5A}$. Let $(x_0, x_1, x_2, x_3, x_4, \nu)$ be a normal $\mathcal{G}U_{5A}$ basis. Since $\tau_{x_1} \cdot \{e, x_0\} = \{e, x_2\}$, $\mathcal{G}\{e, x_2\}$ is isomorphic to $\mathcal{G}\{e, x_0\} \cong \mathcal{G}U_{2A}$. Similarly $\mathcal{G}\{e, x_i\}$ is isomorphic to $\mathcal{G}U_{2A}$ for all $i = 0, 1, 2, 3, 4$.

LEMMA 5.13. *Let* $y_i := \sigma_e(x_i) = e + x_i - 2^2 x_i \cdot e$. *Then* $\mathrm{Gp}\langle \tau_{x_0}, \tau_{y_1} \rangle \cdot \{x_0\} = \{x_0, x_1, x_2, x_3, x_4\}$, *and* $\mathrm{Gp}\langle \tau_{x_0}, \tau_{y_1} \rangle \cdot \{y_1\} = \{y_0, y_1, y_2, y_3, y_4\}$.

PROOF. As in Lemma 5.9, we have $\tau_{x_i} = \tau_{y_i}$ for $i = 0, 1, 2, 3, 4$. Hence $\mathrm{Gp}\langle\tau_{x_0}, \tau_{y_1}\rangle \cdot \{x_0\} = \mathrm{Gp}\langle\tau_{x_0}, \tau_{x_1}\rangle \cdot \{x_0\} = \{x_0, x_1, x_2, x_3, x_4\}$, and $\mathrm{Gp}\langle\tau_{x_0}, \tau_{y_1}\rangle \cdot \{y_1\} = \mathrm{Gp}\langle\tau_{x_0}, \tau_{x_1}\rangle \cdot \{\sigma_e(x_1)\} = \sigma_e \cdot (\mathrm{Gp}\langle\tau_{x_0}, \tau_{x_1}\rangle \cdot \{x_1\}) = \sigma_e \cdot \{x_0, x_1, x_2, x_3, x_4\} = \{y_0, y_1, y_2, y_3, y_4\}$. □

LEMMA 5.14. *For $i, j \in \{0, 1, 2, 3, 4\}$, we have $y_i \neq x_j$.*

PROOF. Suppose $y_i = x_j$ for some i, j. Then $\mathcal{G}\{x_i, y_i\} \cong \mathcal{G}\{x_i, x_j\}$. Since $\mathcal{G}\{x_i, y_i\} \cong \mathcal{G}U_{2A}$ and $\mathcal{G}\{x_i, x_j\} \cong \mathcal{G}U_{5A}$ for $i \neq j$, we must have $i = j$ and $\mathcal{G}\{x_i, x_i\} \cong \mathcal{G}U_{1A}$. It is also absurd. □

Therefore, $\mathcal{G}\{x_0, x_1'\}$ has at least 10 distinct Ising vectors. That is impossible by Theorem 4.4. Hence there is no such Griess algebra.

5.2.7. Case 7. $\mathcal{G}\{x_0, x_1\} \cong \mathcal{G}U_{6A}$. Let $(x_0, x_1, x_2, x_3, x_4,\ x_5, e', u)$ be a normal $\mathcal{G}U_{6A}$ basis of $\mathcal{G}\{x_0, x_1\}$. Since τ_e fixes x_0 and x_1, it also fixes all elements in $\mathcal{G}\{x_0, x_1\}$.

LEMMA 5.15. *Set $y_i := \sigma_e(x_i)$. Then*

(1) $\langle x_0, y_1\rangle, \langle x_0, y_2\rangle \in \{\frac{13}{2^{10}}, \frac{5}{2^{10}}\}$ *and*

(2) $\langle x_0, y_3\rangle \in \{\frac{1}{2^2}, \frac{1}{2^5}, 0\}$.

PROOF. As in (5.9), we have $\tau_{x_i} = \tau_{y_i}$ for $i = 0, 1, 2, 3, 4, 5$. Hence $\mathrm{Gp}\langle\tau_{x_0}, \tau_{y_1}\rangle \cdot \{x_0\} = \{x_0, x_2, x_4\}$, which has 3 elements. So by Theorem 4.4, $\mathcal{G}\{x_0, y_1\}$ is isomorphic to $\mathcal{G}U_{3A}$, $\mathcal{G}U_{3C}$ or $\mathcal{G}U_{6A}$. However, $\mathcal{G}\{x_0, y_1\} \ncong \mathcal{G}U_{3C}$ because $\mathcal{G}\{x_0, y_1\} \supset \mathcal{G}\{x_0, x_2, x_4\} \cong \mathcal{G}U_{3A}$ (or by Case 5).

Similarly, $\mathrm{Gp}\langle\tau_{x_0}\tau_{y_2}\rangle \cdot \{x_0\} = \{x_0, x_2, x_4\}$ and $\mathcal{G}\{x_0, y_2\}$ is also isomorphic to $\mathcal{G}U_{3A}$ or $\mathcal{G}U_{6A}$. Hence by Theorem 4.4, we have

$$\langle x_0, y_1\rangle, \langle x_0, y_2\rangle \in \{\frac{13}{2^{10}}, \frac{5}{2^{10}}\}. \quad (5.2.2)$$

On the other hand, $\mathrm{Gp}\langle\tau_{x_0},\tau_{y_3}\rangle\cdot\{x_0\} = \mathrm{Gp}\langle\tau_{x_0},\tau_{x_3}\rangle\cdot\{x_0\} = \{x_0\}$, which has 1 element. Therefore, $\mathcal{G}\{x_0, y_3\}$ is isomorphic to $\mathcal{G}U_{1A}$, $\mathcal{G}U_{2A}$ or $\mathcal{G}U_{2B}$ by Theorem 4.4 and we have

$$\langle x_0, y_3\rangle \in \{\frac{1}{2^2}, \frac{1}{2^5}, 0\}. \tag{5.2.3}$$

□

LEMMA 5.16. *Let e' be defined as above. Then $\langle e, e'\rangle = \frac{1}{2^2}$ and hence $e = e'$.*

PROOF. By direct calculation, we have

$$\begin{aligned}
&\langle x_0, y_1\rangle \\
=\ & \langle x_0, e + x_1 - 2^2 e\cdot x_1\rangle \\
=\ & \frac{1}{2^5} + \frac{5}{2^{10}} - 2^2\langle e, x_0\cdot x_1\rangle \\
=\ & \frac{1}{2^5} + \frac{5}{2^{10}} - 2^2\left\langle e, \frac{1}{2^5}(x_0 + x_1 - x_2 - x_3 - x_4 - x_5 + e') + \frac{45}{2^{10}}u\right\rangle \text{ by (4.0.3)} \\
=\ & \frac{1}{2^5} + \frac{5}{2^{10}} - 2^2\left(\frac{1}{2^5}\left(\frac{2}{2^5} - \frac{4}{2^5} + \langle e, e'\rangle\right) + \frac{45}{2^{10}}\langle e, u\rangle\right) \\
=\ & \frac{45}{2^{10}} - \frac{1}{2^3}\langle e, e'\rangle + \frac{45}{2^8}\langle e, u\rangle,
\end{aligned}$$

$$\begin{aligned}
\langle x_0, y_3\rangle &= \langle x_0, e + x_3 - 2^2 e\cdot x_3\rangle \\
&= \frac{1}{2^5} + \frac{1}{2^5} - 2^2\langle e, x_0\cdot x_3\rangle \\
&= \frac{1}{4} - 2^2\left\langle e, \frac{1}{2^2}(x_0 + x_3 - e')\right\rangle \\
&= \frac{1}{4} - \frac{1}{2^5} - \frac{1}{2^5} + \langle e, e'\rangle \\
&= \langle e, e'\rangle, \qquad (5.2.4)
\end{aligned}$$

and

$$\begin{aligned}
\langle x_0, y_2\rangle &= \langle x_0, e + x_2 - 2^2 e\cdot x_2\rangle \\
&= \frac{1}{2^5} + \frac{13}{2^{10}} - 2^2\langle e, x_0\cdot x_2\rangle \\
&= \frac{1}{2^5} + \frac{13}{2^{10}} - 2^2\left\langle e, \frac{1}{2^4}(2x_0 + 2x_2 + x_4) - \frac{45}{2^{10}}u\right\rangle \\
&= \frac{45}{2^{10}} - \frac{1}{4}\left(2\cdot\frac{1}{2^5} + 2\cdot\frac{1}{2^5} + \frac{1}{2^5}\right) + \frac{45}{2^8}\langle e, u\rangle \\
&= \frac{5}{2^{10}} - \frac{45}{2^8}\langle e, u\rangle.
\end{aligned}$$

So we have

$$\frac{5}{2^{10}} - \langle x_0, y_2\rangle = \frac{45}{2^8}\langle e, u\rangle = \langle x_0, y_1\rangle - \frac{45}{2^{10}} + \frac{1}{2^3}\langle e, e'\rangle = \langle x_0, y_1\rangle - \frac{45}{2^{10}} + \frac{1}{2^3}\langle x_0, y_3\rangle.$$

Therefore, we have

$$\langle x_0, y_3\rangle = -2^3\langle x_0, y_1\rangle + \frac{50}{2^7} - 2^3\langle x_0, y_2\rangle.$$

Since $\langle x_0, y_1\rangle, \langle x_0, y_2\rangle \in \{\frac{13}{2^{10}}, \frac{5}{2^{10}}\}$,

$$-2^3\langle x_0, y_1\rangle + \frac{50}{2^7} - 2^3\langle x_0, y_2\rangle = \frac{3}{2^4}, \frac{1}{2^2}, \text{ or } \frac{5}{2^4}.$$

Thus, we have $\langle x_0, y_3\rangle = \frac{1}{2^2}$ by (5.2.3) and $\langle e, e'\rangle = \langle x_0, y_3\rangle = \frac{1}{2^2}$ by (5.2.4). That implies $e = e'$ and $x_0 = y_3$ by Remark 2.25. □

In the proof above, we also proved the following.

LEMMA 5.17. *For $i = 0, \ldots, 5$ and $j \equiv i + 3 \mod 6$, we have $y_j = x_i$.*

PROPOSITION 5.18. *The Griess algebra $\mathcal{G}\{e, x_0, x_1\}$ is isomorphic to $\mathcal{G}U_{6A} \cong \mathcal{G}V_{\mathbb{B}(3A)}$.*

PROOF. Since $\mathcal{G}\{e, x_0, x_1\} = \mathcal{G}\{e', x_0, x_1\} = \mathcal{G}\{x_0, x_1\} \cong \mathcal{G}U_{6A}$, we have the desired result (see Figure 2). □

5.2.8. Case 8. $\mathcal{G}\{x_0, x_1\} \cong \mathcal{G}U_{4B}$.

NOTATION 5.19. *Set* $x_2 := \tau_{x_1}(x_0)$, $x_3 := \tau_{x_0}(x_1)$. *Then by the structure of* $\mathcal{G}U_{4B}$ *(Theorem 4.4), we have* $\mathcal{G}\{x_0, x_1\} = \mathrm{Span}_{\mathbb{R}}\{x_0, x_1, x_2, x_3, e'\}$ *where* (x_0, x_1, x_2, x_3, e') *forms a normal* $\mathcal{G}U_{4B}$ *basis.*

LEMMA 5.20. *Let* $b_i := \sigma_e(x_i) = e + x_i - 2^2 e \cdot x_i$. *Then* $\mathrm{Gp}\langle \tau_{x_0}, \tau_{b_1}\rangle \cdot \{x_0, b_1\} = \{x_0, x_2, b_1, b_3\}$. *Therefore,* $\mathcal{G}\{x_0, b_1\} \cong \mathcal{G}U_{4A}$ *or* $\mathcal{G}U_{4B}$ *and* $\langle x_0, b_1\rangle = \frac{1}{2^7}$ *or* $\frac{1}{2^8}$.

PROOF. By Lemma 5.9, we have $\tau_{x_i} = \tau_{b_i}$ for $i = 0, 1, 2, 3$. Hence for $i = 1, 3$, $\tau_{b_i}(x_0) = \tau_{x_i}(x_0) = x_2$, and $b_i \neq x_0, x_2$. Moreover, $\mathrm{Gp}\langle \tau_{x_0}, \tau_{b_1}\rangle \cdot \{x_1\} = \mathrm{Gp}\langle \tau_{x_0}, \tau_{x_1}\rangle \cdot \{x_1\} = \{x_1, x_3\}$, and $\mathrm{Gp}\langle \tau_{x_0}, \tau_{b_1}\rangle \cdot \{b_1\} = \mathrm{Gp}\langle \tau_{x_0}, \tau_{x_1}\rangle \cdot \{\sigma_e(x_1)\} = \sigma_e \cdot (\mathrm{Gp}\langle \tau_{x_0}, \tau_{x_1}\rangle \cdot \{x_1\}) = \sigma_e \cdot \{x_1, x_3\} = \{b_1, b_3\}$. Thus $\mathrm{Gp}\langle \tau_{x_0}, \tau_{b_1}\rangle \cdot \{x_0, b_1\} = \{x_0, x_2, b_1, b_3\}$, which have 4 distinct elements. Hence by Theorem 4.4, we have the lemma. □

LEMMA 5.21. *Let* e' *be defined as in Notation 5.19. Then* $\langle e, e'\rangle \in \{\frac{1}{2^2}, \frac{1}{2^5}, 0\}$.

PROOF. Since τ_e is trivial on $\mathcal{G}$, $\mathrm{Gp}\langle \tau_e, \tau_{e'}\rangle \cdot \{e'\} = \{e'\}$ and $\mathcal{G}\{e, e'\}$ is isomorphic to $\mathcal{G}U_{1A}$, $\mathcal{G}U_{2A}$ or $\mathcal{G}U_{2B}$ by Theorem 4.4. Hence $\langle e, e'\rangle \in \{\frac{1}{2^2}, \frac{1}{2^5}, 0\}$. □

LEMMA 5.22. *Let* e' *and* b_1 *be defined as in Notation 5.19 and Lemma 5.20. Then* $e = e'$ *and* $\langle x_0, b_1\rangle = \frac{1}{2^8}$.

PROOF. By definition,

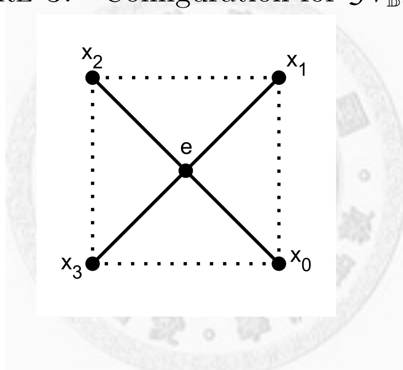
$$\begin{aligned}
\langle x_0, b_1\rangle &= \langle x_0, e + x_1 - 2^2 e \cdot x_1\rangle \\
&= \frac{1}{2^5} + \frac{1}{2^8} - 2^2\langle e, x_0 \cdot x_1\rangle \\
&= \frac{1}{2^5} + \frac{1}{2^8} - 2^2\left\langle e, \frac{1}{2^5}(x_0 + x_1 - x_2 - x_3 + e')\right\rangle \\
&= \frac{1}{2^5} + \frac{1}{2^8} - \frac{1}{2^3}\left(\frac{1}{2^5} + \frac{1}{2^5} - \frac{1}{2^5} - \frac{1}{2^5} + \langle e, e'\rangle\right) \\
&= \frac{9}{2^8} - \frac{1}{2^3}\langle e, e'\rangle.
\end{aligned}$$

Then, by Lemma 5.21, we have $\langle x_0, b_1\rangle = \frac{9}{2^8} - \frac{1}{2^3}\langle e, e'\rangle \in \{\frac{1}{2^8}, \frac{1}{2^5}, \frac{9}{2^8}\}$ and thus $\langle x_0, b_1\rangle = \frac{1}{2^8}$ and $\langle e, e'\rangle = \frac{1}{2^2}$ by Lemma 5.20. It implies $e = e'$ by Remark 2.25. □

PROPOSITION 5.23. *The Griess algebra $\mathcal{G}\{e, x_0, x_1\}$ is isomorphic to $\mathcal{G}U_{4B} \cong \mathcal{G}V_{\mathbb{B}(2C)}$.*

PROOF. Since $e = e'$, we have $\mathcal{G}\{e, x_0, x_1\} = \mathcal{G}\{e', x_0, x_1\} = \mathcal{G}\{x_0, x_1\}$ is isomorphic to $\mathcal{G}U_{4B}$. This Griess algebra is $\mathcal{G}V_{\mathbb{B}(2C)}$. The configuration is given in Figure 3, where three collinear points form a normal $\mathcal{G}U_{2A}$ basis and the 4 vertices of the dotted square and e form a normal $\mathcal{G}U_{4B}$ basis. □

FIGURE 3. Configuration for $\mathcal{G}V_{\mathbb{B}(2C)}$



5.2.9. Case 9. $\mathcal{G}\{x_0, x_1\} \cong \mathcal{G}U_{4A}$.

NOTATION 5.24. *Let $(x_0, x_1, x_2, x_3, \mu_x)$ be a normal $\mathcal{G}U_{4A}$ basis for $\mathcal{G}\{x_0, x_1\}$. Since τ_e is trivial on $\mathcal{G}$, $\tau_e(\mu_x) = \mu_x$ and $\sigma_e(\mu_x)$ is well-defined.*

Set $\mu_y := \sigma_e(\mu_x)$ and let $y_i := \sigma_e(x_i) = e + x_i - 2^2 e \cdot x_i$. We can apply σ_e to the normal $\mathcal{G}U_{4A}$ basis $(x_0, x_1, x_2, x_3, \mu_x)$ to get a new normal $\mathcal{G}U_{4A}$ basis $(y_0, y_1, y_2, y_3, \mu_y)$.

As in Lemma 5.9, we have

$$\sigma_e \tau_{x_i} = \tau_{x_i} \sigma_e \quad \text{and} \quad \tau_{x_i} = \tau_{y_i} \quad \text{for } i = 0, 1, 2, 3. \tag{5.2.5}$$

LEMMA 5.25. *We have $\mathcal{G}\{x_0, y_1\} \cong \mathcal{G}U_{4A}$ and $\mathcal{G}\{y_0, x_1\} \cong \mathcal{G}U_{4A}$.*

PROOF. For $i = 1, 3$, $\tau_{y_i}(x_0) = \tau_{x_i}(x_0) = x_2$, and thus $y_i \neq x_0, x_2$ for $i = 1, 3$. In addition, $\mathrm{Gp}\langle \tau_{x_0}, \tau_{y_1}\rangle \cdot \{x_1\} = \mathrm{Gp}\langle \tau_{x_0}, \tau_{x_1}\rangle \cdot \{x_1\} = \{x_1, x_3\}$, and $\mathrm{Gp}\langle \tau_{x_0}, \tau_{y_1}\rangle \cdot \{y_1\} = \mathrm{Gp}\langle \tau_{x_0}, \tau_{x_1}\rangle \cdot \{\sigma_e(x_1)\} = \sigma_e \cdot (\mathrm{Gp}\langle \tau_{x_0}, \tau_{x_1}\rangle \cdot \{x_1\}) = \sigma_e \cdot \{x_1, x_3\} = \{y_1, y_3\}$. Thus $\mathrm{Gp}\langle \tau_{x_0}, \tau_{y_1}\rangle \cdot \{x_0, y_1\} = \{x_0, x_2, y_1, y_3\}$, which have 4 distinct elements. Hence by Theorem 4.4, $\mathcal{G}\{x_0, y_1\}$ is isomorphic to $\mathcal{G}U_{4A}$ or $\mathcal{G}U_{4B}$ and

$$\langle x_0, y_1\rangle \in \{\frac{1}{2^7}, \frac{1}{2^8}\}.$$

By Norton inequality (Theorem 2.27), we have

$$\langle e, \mu_x\rangle = \frac{1}{4}\langle e \cdot e, \mu_x \cdot \mu_x\rangle \geq \frac{1}{4}\langle e \cdot \mu_x, e \cdot \mu_x\rangle \geq 0.$$

Therefore,

$$\begin{aligned}
\langle x_0, y_1\rangle &= \langle x_0, e + x_1 - 2^2 e \cdot x_1\rangle \\
&= \frac{1}{2^5} + \frac{1}{2^7} - 2^2\langle e, x_0 \cdot x_1\rangle \\
&= \frac{1}{2^5} + \frac{1}{2^7} - 2^2\left\langle e, \frac{1}{2^5}(3x_0 + 3x_1 + x_2 + x_3 - 3\mu_x)\right\rangle \\
&= \frac{1}{2^5} + \frac{1}{2^7} - \frac{1}{2^3}\left(\frac{3}{2^5} + \frac{3}{2^5} + \frac{1}{2^5} + \frac{1}{2^5} - 3\langle e, \mu_x\rangle\right) \\
&= \frac{1}{2^7} + \frac{3}{2^3}\langle e, \mu_x\rangle \\
&\geq \frac{1}{2^7}.
\end{aligned}$$

Hence we have

$$\langle x_0, y_1\rangle = \frac{1}{2^7} \quad \text{and} \quad \langle e, \mu_x\rangle = 0. \tag{5.2.6}$$

Thus $\mathcal{G}\{x_0, y_1\}$ is isomorphic to $\mathcal{G}U_{4A}$. Similarly, $\mathcal{G}\{y_0, x_1\} \cong \mathcal{G}U_{4A}$, also. □

NOTATION 5.26. *By (5.2.5), $\tau_{x_0}(y_1) = \tau_{x_0}\sigma_e(x_1) = \sigma_e\tau_{x_0}(x_1) = \sigma_e(x_3) = y_3$, and $\tau_{y_1}(x_0) = \tau_{x_1}(x_0) = x_2$. Therefore, by the structure of $\mathcal{G}U_{4A}$, there is a conformal vector μ_0 of central charge 1 such that $(x_0, y_1, x_2, y_3, \mu_0)$ forms a normal $\mathcal{G}U_{4A}$ basis for $\mathcal{G}\{y_0, x_1\}$.*

Similarly, there is a conformal vector μ_1 of central charge 1 such that $(y_0, x_1, y_2, x_3, \mu_1)$ forms a normal $\mathcal{G}U_{4A}$ basis for $\mathcal{G}\{y_0, x_1\}$. Note that $\tau_e(\mu_i) = \mu_i$ for all $i \in \{0, 1, x, y\}$ since τ_e is trivial on $\mathcal{G}$.

LEMMA 5.27. *We have $\sigma_e(\mu_1) = \mu_0$, and $\sigma_e(\mu_0) = \mu_1$.*

PROOF. By the structure of $\mathcal{G}U_{4A}$, we have

$$\sigma_e(y_0 \cdot x_1) = \sigma_e(y_0) \cdot \sigma_e(x_1) = x_0 \cdot y_1 = \frac{1}{2^5}(3x_0 + 3y_1 + x_2 + y_3 - 3\mu_0).$$

On the other hand,

$$\begin{aligned} \sigma_e(y_0 \cdot x_1) &= \sigma_e\left(\frac{1}{2^5}(3y_0 + 3x_1 + y_2 + x_3 - 3\mu_1)\right) \\ &= \frac{1}{2^5}(3x_0 + 3y_1 + x_2 + y_3 - 3\sigma_e(\mu_1)). \end{aligned}$$

It implies $\sigma_e(\mu_1) = \mu_0$ and $\sigma_e(\mu_0) = \mu_1$. □

LEMMA 5.28. *We have $\langle e, \mu_i \rangle = 0$ for $i \in \{0, 1, x, y\}$.*

PROOF. To compute $\langle e, \mu_i \rangle$, we use the equation

$$\begin{aligned} \langle e, x_0 \cdot y_1 \rangle &= \left\langle e, \frac{1}{2^5}(3x_0 + 3y_1 + x_2 + y_3 - 3\mu_0)\right\rangle \\ &= \frac{1}{2^5}\left(\frac{3}{2^5} + \frac{3}{2^5} + \frac{1}{2^5} + \frac{1}{2^5} - 3\langle e, \mu_0 \rangle\right) \\ &= \frac{1}{2^7} - \frac{3}{2^5}\langle e, \mu_0 \rangle. \end{aligned}$$

By associative rule,

$$\begin{aligned}\langle e, x_0 \cdot y_1\rangle &= \langle e\cdot x_0, y_1\rangle \\ &= \left\langle \frac{1}{2^2}(e + x_0 - y_0), y_1\right\rangle \\ &= \frac{1}{2^2}\left(\frac{1}{2^5}+\frac{1}{2^7}-\frac{1}{2^7}\right) \\ &= \frac{1}{2^7}.\end{aligned}$$

It implies $\frac{1}{2^7} - \frac{3}{2^5}\langle e, \mu_0\rangle = \frac{1}{2^7}$ and $\langle e, \mu_0\rangle = 0$. Similarly we have $\langle e, \mu_1\rangle = 0$. Combining these results with (5.2.6), we have the lemma. □

LEMMA 5.29. *The subalgebras* $\mathcal{G}\{x_0, y_2\}$, $\mathcal{G}\{x_1, y_3\}$, $\mathcal{G}\{x_2, y_0\}$ *and* $\mathcal{G}\{x_3, y_1\}$ *are isomorphic to* $\mathcal{G}U_{2A}$.

PROOF. We first note that

$$\langle x_0, y_2\rangle = \langle x_0, e + x_2 - 2^2 e\cdot x_2\rangle = \frac{1}{2^5} + 0 - 2^2\langle e, x_0\cdot x_2\rangle = \frac{1}{2^5}.$$

Similarly we have

$$\langle x_0, y_2\rangle = \langle x_1, y_3\rangle = \langle x_2, y_0\rangle = \langle x_3, y_1\rangle = \frac{1}{2^5}. \tag{5.2.7}$$

Hence, $\mathcal{G}\{x_0, y_2\}$, $\mathcal{G}\{x_1, y_3\}$, $\mathcal{G}\{x_2, y_0\}$ and $\mathcal{G}\{x_3, y_1\}$ are isomorphic to $\mathcal{G}U_{2A}$. □

LEMMA 5.30. *For all* $i \in \{0, 1, 2, 3\}, j \in \{x, y, 0, 1\}$, *we have*

$$\langle x_i, \mu_j\rangle = \frac{3}{2^5} \quad \textit{and} \quad \langle y_i, \mu_j\rangle = \frac{3}{2^5}.$$

Proof. We compute

$$\begin{aligned}\langle x_0 \cdot y_0, y_1 \rangle &= \left\langle \frac{1}{2^2}(x_0 + y_0 - e), y_1 \right\rangle \\ &= \frac{1}{2^2}\left(\frac{1}{2^7} + \frac{1}{2^7} - \frac{1}{2^5}\right) \\ &= -\frac{1}{2^8},\end{aligned}$$

and

$$\begin{aligned}\langle x_0 \cdot y_0, y_1 \rangle &= \langle x_0, y_0 \cdot y_1 \rangle \\ &= \left\langle x_0, \frac{1}{2^5}(3y_0 + 3y_1 + y_2 + y_3 - 3\mu_y \right\rangle \\ &= \frac{1}{2^5}\left(\frac{3}{2^5} + \frac{3}{2^7} + \frac{1}{2^5} + \frac{1}{2^7} - 3\langle x_0, \mu_y \rangle\right) \\ &= \frac{5}{2^{10}} - \frac{3}{2^5}\langle x_0, \mu_y \rangle.\end{aligned}$$

Therefore, $\frac{5}{2^{10}} - \frac{3}{2^5}\langle x_0, \mu_y \rangle = -\frac{1}{2^8}$ and $\langle x_0, \mu_y \rangle = \frac{3}{2^5}$.

By the same calculations, we have

$$\langle x_0 \cdot y_0, x_1 \rangle = -\frac{1}{2^8},$$

$$\langle x_0 \cdot y_0, x_1 \rangle = \langle x_0, y_0 \cdot x_1 \rangle = \frac{5}{2^{10}} - \frac{3}{2^5}\langle x_0, \mu_1 \rangle,$$

and then $\langle x_0, \mu_1 \rangle = \frac{3}{2^5}$. The other equality can be proved by the same method. □

Notation 5.31. *Set* $e_0 := \sigma_{x_0}(y_2)$ *and* $e_1 := \sigma_{x_1}(y_3)$.

Proposition 5.32. *The triples* (e_0, x_0, y_2), (e_0, y_0, x_2), (e_1, x_1, y_3), (e_1, y_1, x_3), (e, x_0, y_0), (e, x_1, y_1), (e, x_2, y_2) *and* (e, x_3, y_3) *form normal* $\mathcal{G}U_{2A}$ *bases. Moreover,*

$$\langle x_i, e_0 \rangle = \langle x_i, e_1 \rangle = \langle x_i, e \rangle = \langle y_i, e_0 \rangle = \langle y_i, e_1 \rangle = \langle y_i, e \rangle = \frac{1}{2^5} \quad \textit{for } i \in \{0, 1, 2, 3\}. \tag{5.2.8}$$

Proof. By definition, (e, x_i, y_i) forms a normal $\mathcal{G}U_{2A}$ basis for $i \in \{0,1,2,3\}$. Moreover, (e_0, x_0, y_2) and (e_1, x_1, y_3) also form normal $\mathcal{G}U_{2A}$ bases.

Since $\tau_{x_3}(y_1) = y_1$, $\sigma_{x_3}(y_1)$ is well-defined. Because $x_3 = \tau_{x_0}(x_1)$ and $y_1 = \tau_{x_0}(y_3)$, we have $\sigma_{x_3}(y_1) = \sigma_{\tau_{x_0}(x_1)}(\tau_{x_0}(y_3)) = \tau_{x_0}\sigma_{x_1}(y_3) = \tau_{x_0}(e_1) = e_1$. Therefore, (e_1, y_1, x_3) forms a normal $\mathcal{G}U_{2A}$ basis. Similarly, we can also show that (e_0, y_0, x_2) forms a normal $\mathcal{G}U_{2A}$ basis using $\tau_{x_2}(y_0) = y_0$ and $\sigma_{x_2}(y_0) = e_0$.

It remains to show $\langle x_i, e_0\rangle = \langle y_i, e_0\rangle = \langle x_j, e_1\rangle = \langle y_j, e_1\rangle = \frac{1}{2^5}$ for $i = 1,3$ and $j = 0,2$. Since the calculation is similar, we only prove one case. For example,

$$\begin{aligned}
\langle x_0, e_1\rangle &= \langle x_0, x_1 + y_3 - 2^2 x_1 \cdot y_3\rangle \\
&= \frac{1}{2^7} + \frac{1}{2^7} - 2^2\langle x_0, x_1 \cdot y_3\rangle \\
&= \frac{1}{2^6} - 2^2\langle x_0 \cdot x_1, y_3\rangle \\
&= \frac{1}{2^6} - 2^2\left\langle \frac{1}{2^5}(3x_0 + 3x_1 + x_2 + x_3 - 3\mu_x), y_3\right\rangle \\
&= \frac{1}{2^6} - \frac{1}{2^3}\left(\frac{3}{2^7} + \frac{3}{2^5} + \frac{1}{2^7} + \frac{1}{2^5} - 3\cdot\frac{3}{2^5}\right) \quad \text{by Lemma 5.30} \\
&= \frac{1}{2^5}
\end{aligned}$$

as desired. □

Proposition 5.33. *For each $i \in \{0,1\}$, (e, e_i) is a normal $\mathcal{G}U_{2B}$ basis, i.e.,*

(5.2.9) $$\langle e, e_1\rangle = 0 \quad \textit{and} \quad \langle e, e_0\rangle = 0.$$

PROOF. We note that

$$\begin{aligned}
\langle e, e_1\rangle &= \langle e, x_1 + y_3 - 2^2 x_1 \cdot y_3\rangle \qquad \text{by Proposition 5.32}\\
&= \frac{1}{2^5} + \frac{1}{2^5} - 2^2\langle e \cdot x_1, y_3\rangle\\
&= \frac{1}{2^4} - 2^2\left\langle \frac{1}{2^2}(e + x_1 - y_1), y_3\right\rangle\\
&= \frac{1}{2^4} - \big(\frac{1}{2^5} + \frac{1}{2^5} - 0\big)\\
&= 0.
\end{aligned}$$

Similarly, we also have $\langle e, e_0\rangle = 0$. □

By Lemma 2.24, Lemma 5.27 and (5.2.6), we have

$$\begin{aligned}
e \cdot \mu_x &= 8\langle e, \mu_x\rangle e + \frac{1}{2^2}(\mu_x - \sigma_e(\mu_x))\\
&= \frac{1}{2^2}(\mu_x - \mu_y).
\end{aligned} \tag{5.2.10}$$

Similarly, we have

$$\begin{aligned}
e \cdot \mu_0 &= 8\langle e, \mu_0\rangle e + \frac{1}{2}\Big(\frac{1}{2}(\mu_0 - \sigma_e(\mu_0))\Big)\\
&= \frac{1}{2^2}(\mu_0 - \mu_1)
\end{aligned} \tag{5.2.11}$$

and

$$e \cdot \mu_1 = \frac{1}{2^2}(\mu_1 - \mu_0). \tag{5.2.12}$$

LEMMA 5.34. *We have* $\mu_x = \mu_y$ *and* $\mu_0 = \mu_1$.

Proof. We will compute $\langle \mu_x, \mu_y \rangle$ and $\langle \mu_0, \mu_1 \rangle$. First we note that

$$\begin{aligned} 0 &= \langle e, 2\mu_x \rangle \\ &= \langle e, \mu_x \cdot \mu_x \rangle \\ &= \langle e \cdot \mu_x, \mu_x \rangle \\ &= \langle \frac{1}{2^2}(\mu_x - \mu_y), \mu_x \rangle \\ &= \frac{1}{2^2}(\frac{1}{2} - \langle \mu_x, \mu_y \rangle). \end{aligned}$$

It implies $\langle \mu_x, \mu_y \rangle = \frac{1}{2}$ and thus $\mu_x = \mu_y$ by Remark 2.25 and $\langle \mu_x, \mu_x \rangle = \langle \mu_y, \mu_y \rangle = \frac{1}{2}$.

Similarly, we can also proved that $\langle \mu_0, \mu_1 \rangle = \frac{1}{2}$ and $\mu_0 = \mu_1$. □

Notation 5.35. *Set $\mu := \mu_x = \mu_y$ and $\mu' := \mu_0 = \mu_1$.*

The next proposition is clear from the definition.

Proposition 5.36. *$(x_0, x_1, x_2, x_3, \mu)$, $(y_0, y_1, y_2, y_3, \mu)$, $(x_0, y_1, x_2, y_3, \mu')$, and $(y_0, x_1, y_2, x_3, \mu')$ form normal $\mathcal{G}U_{4A}$ bases.*

Since $\tau_{x_0}(x_1 + x_3) = x_3 + x_1$, $\sigma_{x_0}(x_1 + x_3)$ is well-defined.

Lemma 5.37. *We have*

$$\sigma_{x_0}(x_1 + x_3) = \frac{1}{2^2}(-x_0 + 2x_1 - x_2 + 2x_3 + 3\mu), \tag{5.2.13}$$

and

$$\sigma_{x_0}(y_1 + y_3) = \frac{1}{2^2}(-x_0 + 2y_1 - x_2 + 2y_3 + 3\mu'). \tag{5.2.14}$$

Proof. By (4.0.11),

$$\begin{aligned} x_0 \cdot (x_1 + x_3) &= \frac{1}{2^5}(3x_0 + 3x_1 + x_2 + x_3 - 3\mu) + \frac{1}{2^5}(3x_0 + 3x_3 + x_2 + x_1 - 3\mu) \\ &= \frac{1}{2^4}(3x_0 + 2x_1 + x_2 + 2x_3 - 3\mu). \end{aligned} \tag{5.2.15}$$

By Lemma 2.24, we also have

$$\begin{aligned} x_0 \cdot (x_1 + x_3) &= 8\langle x_0, x_1 + x_3\rangle x_0 + \frac{1}{2^2}((x_1 + x_3) - \sigma_{x_0}(x_1 + x_3)) \\ &= 8\left(\frac{1}{2^7} + \frac{1}{2^7}\right) x_0 + \frac{1}{2^2}((x_1 + x_3) - \sigma_{x_0}(x_1 + x_3)) \\ &= \frac{1}{2^3}(x_0 + 2x_1 + 2x_3 - 2\sigma_{x_0}(x_1 + x_3)). \end{aligned}$$

Hence we have $\frac{1}{2^4}(3x_0 + 2x_1 + x_2 + 2x_3 - 3\mu) = \frac{1}{2^3}(x_0 + 2x_1 + 2x_3 - 2\sigma_{x_0}(x_1 + x_3))$ and get (5.2.13). (5.2.14) can be proved by a similar method. □

LEMMA 5.38. *For all $i \in \{0, 1, 2, 3\}$,*

$$\tau_{x_i}(\mu) = \mu,\ \tau_{y_i}(\mu) = \mu,\ \tau_{x_i}(\mu') = \mu',\ \tau_{y_i}(\mu') = \mu'. \tag{5.2.16}$$

Hence $\sigma_{x_i}(\mu)$, $\sigma_{y_i}(\mu)$, $\sigma_{x_i}(\mu')$ and $\sigma_{y_i}(\mu')$ are well-defined and we have

$$\sigma_{x_0}(\mu) = \frac{1}{2}(x_0 + 2x_1 + x_2 + 2x_3 - \mu). \tag{5.2.17}$$

$$\sigma_{x_0}(\mu') = \frac{1}{2}(x_0 + 2y_1 + x_2 + 2y_3 - \mu'), \tag{5.2.18}$$

$$\sigma_{x_1}(\mu') = \frac{1}{2}(x_1 + 2y_0 + x_3 + 2y_2 - \mu'). \tag{5.2.19}$$

PROOF. First we note that

$$\begin{aligned} \tau_{x_0}(x_0 \cdot (x_1 + x_3)) &= \tau_{x_0}\left(\frac{1}{2^4}(3x_0 + 2x_1 + 2x_3 + x_2 - 3\mu)\right) \qquad \text{by (5.2.15)} \\ &= \frac{1}{2^4}(3x_0 + 2x_3 + 2x_1 + x_2 - 3\tau_{x_0}(\mu)) \end{aligned}$$

and

$$\begin{aligned} \tau_{x_0}(x_0 \cdot (x_1 + x_3)) &= \tau_{x_0}(x_0) \cdot \tau_{x_0}(x_1 + x_3) \\ &= x_0 \cdot (x_3 + x_1) \\ &= \frac{1}{2^4}(3x_0 + 2x_3 + 2x_1 + x_2 - 3\mu) \qquad \text{by (5.2.15).} \end{aligned}$$

Thus, $\frac{1}{2^4}(3x_0+2x_3+2x_1+x_2-3\tau_{x_0}(\mu)) = \frac{1}{2^4}(3x_0+2x_3+2x_1+x_2-3\mu)$ and $\tau_{x_0}(\mu)=\mu$. By similar computations, we have (5.2.16).

We use a similar method to compute $\sigma_{x_0}(\mu)$ and $\sigma_{x_0}(\mu')$.

$$\begin{aligned}\sigma_{x_0}(x_0\cdot(x_1+x_3)) &= \sigma_{x_0}\left(\frac{1}{2^4}(3x_0+2x_1+2x_3+x_2-3\mu)\right) \qquad \text{by (5.2.15)}\\ &= \frac{1}{2^4}\Big(3x_0+\frac{1}{2}(-x_0+2x_1-x_2+2x_3+3\mu)\\ &\quad +x_2-3\sigma_{x_0}(\mu)\Big) \qquad \text{by (5.2.13).}\end{aligned}$$

Moreover,

$$\begin{aligned}\sigma_{x_0}(x_0\cdot(x_1+x_3)) &= \sigma_{x_0}(x_0)\cdot\sigma_{x_0}(x_1+x_3)\\ &= \frac{1}{2^2}x_0\cdot(-x_0+2x_1-x_2+2x_3+3\mu) \qquad \text{by (5.2.13)}\\ &= \frac{1}{2^4}(x_0-2x_1-x_2-2x_3+3\mu) \qquad \text{by (4.0.2, 4.0.11, 4.0.12).}\end{aligned}$$

Hence we have

$$\frac{1}{2^4}\left(3x_0+\frac{1}{2}(-x_0+2x_1-x_2+2x_3+3\mu)+x_2-3\sigma_{x_0}(\mu)\right) = \frac{1}{2^4}(x_0-2x_1-x_2-2x_3+3\mu)$$

and obtain (5.2.17). (5.2.18) and (5.2.19) can be obtained by the same method. □

LEMMA 5.39. *Let e_0 and e_1 be defined as in Notation 5.31. Then $\mathcal{G}\{e_0,e_1\}$ is isomorphic to $\mathcal{G}U_{1A}$, i.e., $e_0=e_1$.*

PROOF. First, we note by Proposition 5.32 that

$$\begin{aligned}\tau_{e_1}(e_0) &= \tau_{e_1}(x_0+y_2-2^2x_0\cdot y_2)\\ &= x_0+y_2-2^2x_0\cdot y_2\\ &= e_0.\end{aligned}$$

Thus, by Theorem 4.4, $\mathcal{G}\{e_0,e_1\}$ is isomorphic to either $\mathcal{G}U_{1A}$, $\mathcal{G}U_{2A}$ or $\mathcal{G}U_{2B}$.

Case 1. Suppose $\mathcal{G}\{e_1, e_0\}$ is isomorphic to $\mathcal{G}U_{2A}$. By Proposition 5.32, $\mathcal{G}\{e_1, x_0\}$ and $\mathcal{G}\{e_1, y_2\}$ are both isomorphic to $\mathcal{G}U_{2A}$.

Since (e_0, x_0, y_2) also forms a normal $\mathcal{G}U_{2A}$ basis by Proposition 5.32, such a Griess algebra doesn't exist by Lemma 5.3.

Case 2. Suppose $\mathcal{G}\{e_0, e_1\} \cong \mathcal{G}U_{2B}$. Then $\langle e_0, e_1\rangle = 0$. By Proposition 5.32, $\mathcal{G}\{e_0, x_1\}$ and $\mathcal{G}\{e_0, x_3\}$ are both isomorphic to $\mathcal{G}U_{2A}$. Set $x_{10} := \sigma_{e_0}(x_1)$. Then we have

$$\langle x_0, x_{10}\rangle = \langle \sigma_{e_0}(x_0), \sigma_{e_0}(x_{10})\rangle = \langle y_2, x_1\rangle = \frac{1}{2^7}, \tag{5.2.20}$$

$$\langle x_2, x_{10}\rangle = \langle \sigma_{e_0}(x_2), \sigma_{e_0}(x_{10})\rangle = \langle y_0, x_1\rangle = \frac{1}{2^7}. \tag{5.2.21}$$

By Proposition 5.33 and (5.2.19), we also have

$$\langle y_1, x_{10}\rangle = \langle \sigma_{x_1}(y_1), \sigma_{x_1}(x_{10})\rangle = \langle e, e_0\rangle = 0, \tag{5.2.22}$$

$$\langle y_3, x_{10}\rangle = \langle \sigma_{x_1}(y_3), \sigma_{x_1}(x_{10})\rangle = \langle e_1, e_0\rangle = 0, \tag{5.2.23}$$

$$\begin{aligned} \langle \mu', x_{10}\rangle &= \langle \sigma_{x_1}(\mu'), \sigma_{x_1}(x_{10})\rangle = \left\langle \frac{1}{2}(x_1 + 2y_0 + x_3 + 2y_2 - \mu'), e_0 \right\rangle \\ &= \frac{3}{2^5}. \end{aligned} \tag{5.2.24}$$

Since $x_{10} = \sigma_{e_0}(x_1) = \sigma_{x_1}(e_0)$ and $y_1 = \sigma_e(x_1) = \sigma_{x_1}(e)$, we have

$$\langle y_1, x_{10}\rangle = (\sigma_{x_1}(e), \sigma_{x_1}(e_0)\rangle = \langle e, e_0\rangle = 0 \qquad \text{by (5.2.9)}.$$

Thus $\mathcal{G}\{y_1, x_{10}\} \cong \mathcal{G}U_{2B}$ and hence $x_{10} \cdot y_1 = 0$. Therefore,

$$\begin{aligned}
0 &= \langle 0, \mu' \rangle = \langle x_{10} \cdot y_1, \mu' \rangle = \langle x_{10}, y_1 \cdot \mu' \rangle \\
&= \left\langle x_{10}, \frac{1}{2^3}(5y_1 - 2x_0 - 2x_2 - y_3 + 3\mu') \right\rangle \qquad \text{by (4.0.12)} \\
&= \frac{1}{2^8}(-1 + 3 \cdot 2^5 \langle x_{10}, \mu' \rangle) \qquad \text{by (5.2.20, 5.2.21, 5.2.22, 5.2.23)} \\
&= \frac{1}{2^8}(-1 + 3 \cdot 2^5 \cdot \frac{3}{2^5}) \qquad \text{by (5.2.24)} \\
&= \frac{1}{2^5},
\end{aligned}$$

which is a contradiction. Therefore, $\mathcal{G}\{e_0, e_1\} \ncong \mathcal{G}U_{2B}$ and hence $\mathcal{G}\{e_0, e_1\} \cong \mathcal{G}U_{1A}$ is the only possible case. □

NOTATION 5.40. *Set $e' := e_1 = e_0$. Then $e' = \sigma_{x_0}(y_2) = \sigma_{x_1}(y_3)$ by Notation 5.31.*

LEMMA 5.41. *Let μ, μ' be defined as in Notation 5.35. Then*

$$\langle e', \mu \rangle = \langle e', \mu' \rangle = 0 \quad \textit{and} \quad e' \cdot \mu = e' \cdot \mu' = 0 \tag{5.2.25}$$

PROOF. Recall that $e' = \sigma_{x_0}(y_2) = \sigma_{x_1}(y_3)$. Then by the same argument as in Lemma 5.27, one can show that $\sigma_{e'}(\mu) = \mu$ and $\sigma_{e'}(\mu') = \mu'$.

To determine $\langle e', \mu' \rangle$, we compute $\langle e', x_0 \cdot y_3 \rangle$ in two different ways. First, we have

$$\langle e', x_0 \cdot y_3 \rangle = \left\langle e', \frac{1}{2^5}(3x_0 + 3y_3 + x_2 + y_1 - 3\mu') \right\rangle = \frac{1}{2^7} - \frac{3}{2^5}\langle e', \mu' \rangle.$$

By associative rule,

$$\langle e', x_0 \cdot y_3 \rangle = \langle e' \cdot x_0, y_3 \rangle = \left\langle \frac{1}{2^2}(e' + x_0 - y_2), y_3 \right\rangle = \frac{1}{2^7}.$$

Thus, we have $\frac{1}{2^7} - \frac{3}{2^5}\langle e', \mu' \rangle = \frac{1}{2^7}$ and $\langle e', \mu' \rangle = 0$. By the similar argument, one can also show that $\langle e', \mu \rangle = 0$ by using $\langle e', x_0 \cdot x_1 \rangle$.

By Lemma 2.24, we have

$$e' \cdot \mu' = 8\langle e', \mu' \rangle e' + \frac{1}{2^2}(\mu' - \sigma_{e'}(\mu')) = 0$$

since $\langle e', \mu' \rangle = 0$ and $\sigma_{e'}(\mu') = \mu'$. Similarly, $e' \cdot \mu = 0$ also. $\square$

PROPOSITION 5.42. *We have* $\langle \mu, \mu' \rangle = 0$ *and* $\mu \cdot \mu' = 0$. *Moreover, we have the relation*

$$x_0 + x_1 + x_2 + x_3 + y_0 + y_1 + y_2 + y_3 - e - e' - \frac{3}{2}\mu' - \frac{3}{2}\mu = 0. \tag{5.2.26}$$

PROOF. To compute $\langle \mu, \mu' \rangle$, we note that

$$\begin{aligned}
\langle x_0, x_1 \cdot \mu' \rangle &= \left\langle x_0, \frac{1}{2^3}(5x_1 - 2y_0 - 2y_2 - x_3 + 3\mu') \right\rangle && \text{by (4.0.12)} \\
&= \frac{1}{2^3}\Big(5 \cdot \frac{1}{2^7} - 2 \cdot \frac{1}{2^5} - 2 \cdot \frac{1}{2^5} - \frac{1}{2^7} + 3 \cdot \frac{3}{2^5}\Big) && \text{by Propositions 5.32, 5.36} \\
&= \frac{3}{2^7}.
\end{aligned}$$

Moreover,

$$\langle x_0, x_1 \cdot \mu' \rangle = \langle x_0 \cdot x_1, \mu' \rangle = \left\langle \frac{1}{2^5}(3x_0 + 3x_1 + x_2 + x_3 - 3\mu), \mu' \right\rangle = \frac{3}{2^7}(1 - 4\langle \mu, \mu' \rangle).$$

Hence we have $\frac{3}{2^7}(1 - 4\langle \mu, \mu' \rangle) = \frac{3}{2^7}$ and $\langle \mu, \mu' \rangle = 0$.

To get the relation of $\{e, e', x_0, x_1, x_2, x_3, y_0, y_1, y_2, y_3, \mu, \mu'\}$, we use

$$\begin{aligned}
\sigma_{x_0}((x_1 + x_3) \cdot e) &= \sigma_{x_0}\left(\frac{1}{2^2}(x_1 + e - y_1) + \frac{1}{2^2}(x_3 + e - y_3)\right) \\
&= \frac{1}{2^2}(2\sigma_{x_0}(e) + \sigma_{x_0}(x_1 + x_3) - \sigma_{x_0}(y_1 + y_3)) \\
&= \frac{1}{2^2}\Big(2y_0 + \frac{1}{4}(-x_0 + 2x_1 - x_2 + 2x_3 + 3\mu) \\
&\quad - \frac{1}{2}(-x_0 + 2y_1 - x_2 + 2y_3 + 3\mu')\Big) \qquad \text{by(5.2.13, 5.2.14)} \\
&= \frac{1}{2^4}(8y_0 + 2x_1 + 2x_3 - 2y_1 - 2y_3 + 3\mu - 3\mu')
\end{aligned}$$

and

$$\begin{aligned}
\sigma_{x_0}((x_1+x_3)\cdot e) &= \sigma_{x_0}(x_1+x_3)\cdot\sigma_{x_0}(e) \\
&= \frac{1}{2^2}(-x_0+2x_1-x_2+2x_3+3\mu)\cdot y_0 \qquad \text{by (5.2.13)} \\
&= \frac{1}{2^2}\Big(\frac{-1}{2^2}(x_0+y_0-e)+\frac{2}{2^4}(3x_1+3y_0+x_3+y_2-3\mu') \\
&\quad +\frac{-1}{2^2}(x_2+y_0-e')+\frac{2}{2^4}(3x_3+3y_0+x_1+y_2-3\mu') \\
&\quad +\frac{3}{2^3}(5x_0-2x_1-2x_3-x_2+3\mu)\Big) \\
&= \frac{1}{2^5}(-2x_0+2x_1-2x_2+2x_3+14y_0-6y_1-2y_2-6y_3 \\
&\quad +2e+2e'-3\mu'+9\mu).
\end{aligned}$$

Hence we have $\frac{1}{2^5}(-2x_0+2x_1-2x_2+2x_3+14y_0-6y_1-2y_2-6y_3+2e+2e'-3\mu'+9\mu)=\frac{1}{2^4}(8y_0+2x_1+2x_3-2y_1-2y_3+3\mu-3\mu')$ and get the relation (5.2.26).

To obtain $\mu\cdot\mu'=0$, we simply multiply (5.2.26) by μ and simplify it. $\square$

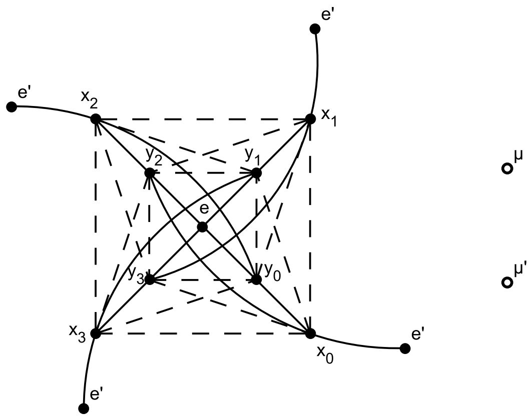
To summarize, we show that $\mathcal{G}=\mathrm{Span}_{\mathbb{R}}\{e,e',x_0,x_1,x_2,x_3,y_0,y_1,y_2,y_3,\mu,\mu'\}$ is closed under multiplication with the relation

$$x_0+x_1+x_2+x_3+y_0+y_1+y_2+y_3-e-e'-\frac{3}{2}\mu-\frac{3}{2}\mu'=0.$$

This algebra is isomorphic to $\mathcal{G}V_{\mathbb{B}(4B)}$. The structure is summarized in Figure 4. Three points joined by a solid line (curve) form a normal $\mathcal{G}U_{2A}$ basis while the 4 vertices of a dotted square form a normal $\mathcal{G}U_{4A}$ basis with μ and the 4 vertices of a dotted diamond form a normal $\mathcal{G}U_{4A}$ basis with μ'.

Remark 5.43. To see $\{e,e',x_0,x_1,x_2,x_3,y_0,y_1,y_2,y_3,\mu\}$ is linear independent, one can compute the determinant of their Gram matrix. By computer, we verify that the determinant is $\frac{3^6}{2^{32}}\neq 0$ and hence $\mathcal{G}V_{\mathbb{B}(4B)}$ have dimension 11.

FIGURE 4. Configuration for $\mathcal{G}V_{\mathbb{B}(4B)}$



Therefore, there are only five possible structures for $\mathcal{G}\{e, x_0, x_1\}$ and we have proved Theorem 5.4.

CHAPTER 6

Griess algebras generated by two $3A$-algebras with a common axis

In this Chapter, we study Griess algebras generated by two pairs of Ising vectors (a_0, a_1) and (b_0, b_1) such that each pair generates a $3A$-algebra U_{3A} and their intersection contains the W_3-algebra $\mathcal{W}(4/5) \cong L(4/5, 0) \oplus L(4/5, 3)$. We show that there are only 3 possibilities, up to isomorphisms and they are isomorphic to the Griess algebras of the VOA $V_{F(1A)}$, $V_{F(2A)}$, and $V_{F(3A)}$ constructed in [**HLY2**].

In addition to the symmetry of τ and σ involutions, we need the help of another order 3 automorphism associated to the W_3-algebra $\mathcal{W}(4/5)$.

6.1. An order 3 automorphism induced by $\mathcal{W}(4/5)$

We will describe an order 3 automorphism associated to the W_3-algebra $\mathcal{W}(4/5) = L(\frac{4}{5}, 0) \oplus L(\frac{4}{5}, 3)$ over the real field $\mathbb{R}$.

Let $L(\frac{4}{5}, 0)_{\mathbb{C}}$ be the Virasoro VOA of central charge 4/5 over the complex field. It is known (see [**KMY**] and [**LLY**]) that the sum $\mathcal{W}_{\mathbb{C}}(4/5) = L(\frac{4}{5}, 0)_{\mathbb{C}} \oplus L(\frac{4}{5}, 3)_{\mathbb{C}}$ has a unique structure of a simple VOA over $\mathbb{C}$. This VOA is rational and has exactly 6 irreducible modules, namely,

$$W_{\mathbb{C}}(0) = L(\frac{4}{5}, 0)_{\mathbb{C}} \oplus L(\frac{4}{5}, 3)_{\mathbb{C}}, \qquad W_{\mathbb{C}}(2/5) = L(\frac{4}{5}, \frac{2}{5})_{\mathbb{C}} \oplus L(\frac{4}{5}, \frac{7}{5})_{\mathbb{C}},$$
$$W_{\mathbb{C}}(2/3, +) = L(\frac{4}{5}, \frac{2}{3})_{\mathbb{C}}, \qquad W_{\mathbb{C}}(2/3, -) = L(\frac{4}{5}, \frac{2}{3})_{\mathbb{C}},$$
$$W_{\mathbb{C}}(1/15, +) = L(\frac{4}{5}, \frac{1}{15})_{\mathbb{C}}, \qquad W_{\mathbb{C}}(1/15, -) = L(\frac{4}{5}, \frac{1}{15})_{\mathbb{C}}.$$

Its fusion rules has a $\mathbb{Z}_3$-symmetry and one can define an automorphism as follows.

Theorem 6.1 ([**Mi2**], Theorem 5.1). *Let $V_{\mathbb{C}}$ be a VOA over $\mathbb{C}$ containing a sub-VOA $X_{\mathbb{C}}$ isomorphic to $\mathcal{W}_{\mathbb{C}}(4/5)$. Then a linear endomorphism $g_{X_{\mathbb{C}}}$ of $V_{\mathbb{C}}$ defined by*

$$g_{X_{\mathbb{C}}} := \begin{cases} 1 & \text{on the isotypic components of } W_{\mathbb{C}}(0) \text{ and } W_{\mathbb{C}}(2/5) \\ e^{2\pi i/3} & \text{on the isotypic components of } W_{\mathbb{C}}(2/3,+) \text{ and } W_{\mathbb{C}}(1/15,+) \\ e^{4\pi i/3} & \text{on the isotypic components of } W_{\mathbb{C}}(2/3,-) \text{ and } W_{\mathbb{C}}(1/15,-) \end{cases}$$

is an automorphism of $V_{\mathbb{C}}$.

In [**Mi2**], the real form of $\mathcal{W}_{\mathbb{C}}(4/5)$, i.e. a real sub-VOA $\mathcal{W}_{\mathbb{R}}^+$ such that $\mathcal{W}_{\mathbb{R}}^+ \otimes \mathbb{C} = \mathcal{W}_{\mathbb{C}}(4/5)$, has been studied.

Proposition 6.2 ([**Mi2**], Theorem 6.1). *There is a unique real sub-VOA $\mathcal{W}_{\mathbb{R}}^+$ of $\mathcal{W}_{\mathbb{C}}(4/5)$ which possesses a positive definite invariant bilinear form over $\mathbb{R}$ and $\mathcal{W}_{\mathbb{R}}^+ \otimes \mathbb{C} = \mathcal{W}_{\mathbb{C}}(4/5)$. This VOA $\mathcal{W}_{\mathbb{R}}^+$ is rational.*

The automorphism defined in Theorem 6.1 restricts to $V_{\mathbb{R}}$ as following.

Theorem 6.3. *[[**Mi2**], Theorem 6.2] Assume that a VOA $V_{\mathbb{R}}$ over $\mathbb{R}$ contains a sub-VOA $X \cong \mathcal{W}_{\mathbb{R}}^+$. Then the automorphism $g_{\mathbb{C}X} \in Aut(\mathbb{C}V_{\mathbb{R}})$ defined by $\mathbb{C}X$ as in Theorem 6.1 keeps $V_{\mathbb{R}}$ invariant. In particular, $g_X = g_{\mathbb{C}X}|_{V_{\mathbb{R}}}$ is an automorphism of $V_{\mathbb{R}}$.*

Now suppose $U \cong U_{3A}$ is contained in a real VOA V satisfying Assumption 1. Let (a_0, a_1, a_2, μ) be a normal $\mathcal{G}U_{3A}$ basis of U. Then U contains a unique sub-VOA X isomorphic to $\mathcal{W}_{\mathbb{R}}^+$ (see [**LYY2, SY**]). In this case, the Virasoro element of X is μ. By the theorem above, $g = g_{\mathbb{C}X}|_{V_{\mathbb{R}}}$ defines an order 3 automorphism on V and U.

Lemma 6.4 ([**LYY2, SY**]). *Let (a_0, a_1, a_2, μ) be a normal $\mathcal{G}U_{3A}$ basis of U and let $g = g_{\mathbb{C}X}|_{V_{\mathbb{R}}}$. Then $\tau_{a_0}\tau_{a_1} = g$ or g^{-1}.*

6.2. Main setting

Let V be a VOA satisfying Assumption 1. Let $U \cong U_{3A}$ and $U' \cong U_{3A}$ be sub-VOA of V. We further assume that $U \cap U'$ contains a sub-VOA X isomorphic to $\mathcal{W}_{\mathbb{R}}^{+}$. Let μ be the Virasoro element of $\mathcal{W}_{\mathbb{R}}^{+}$. Then μ generates a sub-VOA isomorphic to $L(4/5,0)_{\mathbb{R}}$. Since $U \cong U_{3A}$, the Griess algebra of U is of dimension 4 and $\mathcal{G}U = \mathrm{Span}\{a_0, a_1, a_2, \mu\}$, where a_0, a_1, a_2 are the three distinct Ising vectors in $\mathcal{G}U$. Similarly, $\mathcal{G}U' = \mathrm{Span}\{b_0, b_1, b_2, \mu\}$, where b_0, b_1, b_2 are the three distinct Ising vectors in $\mathcal{G}U'$. By Lemma 6.4, we may assume that

$$\tau_{a_0}\tau_{a_1} = \tau_{b_0}\tau_{b_1} = g_X \tag{6.2.1}$$

by reindexing if necessary.

LEMMA 6.5. *Let $g = g_X$ be defined as in Theorem 6.3. Then*

$$\tau_{a_i} g = g^{-1}\tau_{a_i} \tag{6.2.2}$$

and g commutes with $\tau_{a_i}\tau_{b_j}$ for any $i, j \in \{0,1,2\}$.

PROOF. Since $g = g_X = \tau_{a_0}\tau_{a_1} = \tau_{b_0}\tau_{b_1}$, both τ_{a_i} and τ_{b_j} invert g. Hence, we have

$$\tau_{a_i}\tau_{b_j} g = \tau_{a_i} g^{-1}\tau_{b_j} = g\tau_{a_i}\tau_{b_j}$$

as desired. □

In [**HLY2**], McKay's E_6-observation and the Fischer group Fi_{24} were studied. Along with other results, three VOA $V_{F(1A)}$, $V_{F(2A)}$, and $V_{F(3A)}$ generated by two $3A$ algebras were constructed. We will denote their Griess algebras by $\mathcal{G}V_{F(1A)}$, $\mathcal{G}V_{F(2A)}$, and $\mathcal{G}V_{F(3A)}$ respectively. With Assumption 1, we will show that these three cases exhaust all possibilities for Griess algebras generated by $\mathcal{G}U$ and $\mathcal{G}U'$. The following is our main theorem.

6.3. The second main theorem

THEOREM 6.6. *Let V be a VOA over $\mathbb{R}$ satisfying Assumption 1. Let $U \cong U_{3A}$ and $U' \cong U_{3A}$ be sub-VOA of V such that $U \cap U'$ contains a sub-VOA isomorphic to $\mathcal{W}_{\mathbb{R}}^{+}$. Let (a_0, a_1, a_2, μ) and (b_0, b_1, b_2, μ) be normal $\mathcal{G}U_{3A}$ bases of $\mathcal{G}U$ and $\mathcal{G}U'$ respectively and let $\mathcal{G}$ be the sub-Griess algebra generated by $\mathcal{G}U$ and $\mathcal{G}U'$. Then $\mathcal{G}$ is isomorphic to one of the following 3 structures.*

(1) $\mathcal{G} \cong \mathcal{G}V_{F(1A)} \cong \mathcal{G}U_{3A}$. *In this case,* $\{a_0, a_1, a_2\} = \{b_0, b_1, b_2\}$.

(2) $\mathcal{G} \cong \mathcal{G}V_{F(2A)} \cong \mathcal{G}U_{6A}$.

(3) $\mathcal{G} \cong \mathcal{G}V_{F(3A)}$. *In this case,* $\dim \mathcal{G} = 12$ *and it is spanned by* 9 *Ising vectors* $x_{i,j}$, $i, j \in \mathbb{Z}_3$ *and* 4 *Virasoro vectors* $\mu_{0,1} = \mu$, $\mu_{1,0}$, $\mu_{1,1}$ *and* $\mu_{1,2}$ *of central charge* $4/5$ *satisfying the relation*

$$32 \sum_{i,j \in \mathbb{Z}_3} x_{i,j} - 45(\mu_{0,1} + \mu_{1,0} + \mu_{1,1} + \mu_{1,2}) = 0.$$

Moreover, $(x_{i_0,j_0}, x_{i_1,j_1} x_{i_2,j_2}, \mu_{i,j})$ *forms a normal* $\mathcal{G}U_{3A}$ *basis if and only if*

$$\begin{cases} (i_0, j_0) + (i_1, j_1) + (i_2, j_2) \equiv (0, 0) \pmod 3, \\ (i_1, j_1) - (i_0, j_0) \equiv \pm(i, j) \pmod 3. \end{cases}$$

By Theorem 4.4, there are 9 possible structures for $\mathcal{G}\{a_0, b_0\}$. We will prove our main theorem by analyzing these 9 cases in details.

6.3.1. Case1: $\mathcal{G}\{a_0, b_0\} \cong \mathcal{G}U_{1A}$.

PROPOSITION 6.7. *Suppose $a_i = b_j$ for some $i, j \in \{0, 1, 2\}$. Then $\{a_0, a_1, a_2\} = \{b_0, b_1, b_2\}$ and $\mathcal{G} \cong \mathcal{G}U_{3A}$. In particular, $\mathcal{G} \cong \mathcal{G}U_{3A}$ if $\mathcal{G}\{a_0, b_0\} \cong \mathcal{G}U_{1A}$.*

PROOF. Without loss, we may assume $a_0 = b_0$. Then

$$\begin{aligned}\langle a_0 \cdot \mu, b_1 \rangle &= \langle \frac{2}{3^2}(2a_0 - a_1 - a_2) + \frac{5}{2^4}\mu, b_1 \rangle \qquad \text{by (4.0.4)} \\ &= \frac{2}{3^2}\big(2 \cdot \frac{13}{2^{10}} - \langle a_1, b_1 \rangle - \langle a_2, b_1 \rangle\big) + \frac{5}{2^4} \cdot \frac{1}{2^4} \qquad \text{by (4.0.6).}\end{aligned}$$

On the other hand,

$$\begin{aligned}\langle a_0 \cdot \mu, b_1 \rangle &= \langle a_0, \mu \cdot b_1 \rangle \qquad \text{by (2.7.4)} \\ &= \langle b_0, \frac{2}{3^2}(2b_1 - b_0 - b_2) + \frac{5}{2^4}\mu \rangle \qquad \text{by (4.0.4)} \\ &= \frac{2}{3^2}\big(2 \cdot \frac{13}{2^{10}} - \frac{1}{2^2} - \frac{13}{2^{10}}\big) + \frac{5}{2^4} \cdot \frac{1}{2^4} \qquad \text{by (4.0.6).}\end{aligned}$$

Hence we have

$$\langle a_1, b_1 \rangle + \langle a_2, b_1 \rangle = \frac{267}{2^{10}},$$

which implies $\max\{\langle a_1, b_1 \rangle, \langle a_2, b_1 \rangle\} \geq \frac{1}{2} \cdot \frac{267}{2^{10}} > \frac{1}{2^5}$. Thus, we have $b_1 = a_1$ or $b_1 = a_2$ since by Theorem 4.4 and Remark 4.5, $\langle a_i, b_j \rangle \leq \frac{1}{2^5}$ if $a_i \neq b_j$. In either case, we have $\{a_0, a_1, a_2\} = \{b_0, b_1, b_2\}$ and $\mathcal{G}$ is isomorphic to $\mathcal{G}U_{3A}$. □

6.3.2. Case2: $\mathcal{G}\{a_0, b_0\} \cong \mathcal{G}U_{2A}$. In this case, set $c_0 = \sigma_{a_0}(b_0)$. Then by (4.0.1), we have $\mathcal{G}\{a_0, b_0\} = \text{Span}\{a_0, b_0, c_0\}$,

(6.3.1) $$a_0 \cdot b_0 = \frac{1}{2^2}(a_0 + b_0 - c_0) \quad \text{and} \quad \langle a_0, b_0 \rangle = \frac{1}{2^5}.$$

PROPOSITION 6.8. *Suppose $\mathcal{G}\{a_0, b_0\} \cong \mathcal{G}U_{2A}$. Then $\mathcal{G} = \mathcal{G}\{a_0, b_1\} = \mathcal{G}\{a_0, b_2\} \cong \mathcal{G}U_{6A}$.*

PROOF. We will first calculate the values of $\langle a_0, b_j \rangle$ for $j = 1, 2$. By (6.3.1) and (4.0.6), we have

$$\langle a_0 \cdot b_0, b_1 \rangle = \langle \frac{1}{2^2}(a_0 + b_0 - c_0), b_1 \rangle = \frac{1}{2^2}\Big(\langle a_0, b_1 \rangle + \frac{13}{2^{10}} - \langle c_0, b_1 \rangle\Big),$$

and

$$\langle a_0 \cdot b_0, b_1\rangle = \langle a_0, b_0 \cdot b_1\rangle = \langle a_0, \frac{1}{2^4}(2b_0 + 2b_1 + b_2) - \frac{135}{2^{10}}\mu\rangle$$
$$= \frac{2}{2^4}\cdot\frac{1}{2^5} + \frac{2}{2^4}\langle a_0, b_1\rangle + \frac{1}{2^4}\langle a_0, b_2\rangle - \frac{135}{2^{10}}\cdot\frac{1}{2^4}.$$

Combining these two equations, we obtain

(6.3.2) $$123 = 2^{10}\big(-2\langle a_0, b_1\rangle + \langle a_0, b_2\rangle + 2^2\langle c_0, b_1\rangle\big).$$

and

(6.3.3) $$\langle c_0, b_1\rangle = \frac{123}{2^{12}} + \frac{1}{2}\langle a_0, b_1\rangle - \frac{1}{2^2}\langle a_0, b_2\rangle.$$

Since by Theorem 4.4,

(6.3.4) $$\langle a_0, b_1\rangle, \langle a_0, b_2\rangle, \langle c_0, b_1\rangle \in \left\{\frac{1}{2^2}, \frac{1}{2^5}, \frac{13}{2^{10}}, \frac{1}{2^7}, \frac{3}{2^9}, \frac{5}{2^{10}}, \frac{1}{2^8}, 0\right\},$$

we have

$$\langle c_0, b_1\rangle = \frac{123}{2^{12}} + \frac{1}{2}\langle a_0, b_1\rangle - \frac{1}{2^2}\langle a_0, b_2\rangle \le \frac{123}{2^{12}} + \frac{1}{2}\cdot\frac{1}{2^2} < \frac{1}{2^2}.$$

Hence $c_0 \neq b_1$ and $\langle c_0, b_1\rangle \le \frac{1}{2^5}$.

We also note that $a_0 \neq b_1$ and $a_0 \neq b_2$; otherwise, $\{a_0, a_1, a_2\} = \{b_0, b_1, b_2\}$ by Proposition 6.7 and $\mathcal{G}\{a_0, b_0\} \not\cong \mathcal{G}U_{2A}$. Therefore, $\langle a_0, b_1\rangle \le \frac{1}{2^5}$ and $\langle a_0, b_2\rangle \le \frac{1}{2^5}$.

Now by (6.3.3), we have

$$\langle c_0, b_1\rangle = \frac{123}{2^{12}} + \frac{1}{2}\langle a_0, b_1\rangle - \frac{1}{2^2}\langle a_0, b_2\rangle \ge \frac{123}{2^{12}} - \frac{1}{2^2}\cdot\frac{1}{2^5} = \frac{91}{2^{12}} > \frac{13}{2^{10}},$$

and hence

(6.3.5) $$\langle c_0, b_1\rangle = \frac{1}{2^5}.$$

Therefore by (6.3.2), we have

(6.3.6) $$2^{11}\langle a_0, b_1\rangle = 2^{10}\langle a_0, b_2\rangle + 5.$$

Note that $2^{11}\langle a_0, b_1\rangle$ is an even integer, so $2^{10}\langle a_0, b_2\rangle$ is an odd integer and hence $\langle a, b_2\rangle = \frac{5}{2^{10}}$ or $\frac{13}{2^{10}}$ by (6.3.4). If $\langle a_0, b_2\rangle = \frac{13}{2^{10}}$, then $\langle a_0, b_1\rangle = \frac{9}{2^{10}}$ which is impossible. Hence, we have $\langle a_0, b_2\rangle = \frac{5}{2^{10}}$ and $\langle a_0, b_1\rangle = \frac{5}{2^{10}}$. That means $\mathcal{G}\{a_0, b_1\} \cong \mathcal{G}\{a_0, b_2\} \cong \mathcal{G}U_{6A}$ and $\mathcal{G}\{c_0, b_1\} \cong \mathcal{G}U_{2A}$.

Claim: $\mathcal{G} = \mathcal{G}\{a_0, b_1\} \cong \mathcal{G}U_{6A}$.

Let $(a_0, b_1, x_2, x_3, x_4, x_5, e, \mu')$ be the normal $\mathcal{G}U_{6A}$ basis for $\mathcal{G}\{a_0, b_1\}$. We will show that $x_3 = b_0$, $x_5 = b_2$, $\{x_2, x_4\} = \{a_1, a_2\}$, $e = c_0$, $\mu' = \mu$ and $\mathcal{G} = \mathcal{G}\{a_0, b_1\} \cong \mathcal{G}U_{6A}$.

Since $\mathcal{G}\{c_0, a_0\} \cong \mathcal{G}\{c_0, b_0\} \cong \mathcal{G}\{c_0, a_1\} \cong \mathcal{G}\{c_0, b_1\} \cong \mathcal{G}U_{2A}$ and $\mathcal{G}$ is generated by a_0, a_1, b_0, b_1, the map σ_{c_0} is well-defined on $\mathcal{G}$. Moreover,

(6.3.7)
$$\tau_{b_0}\sigma_{c_0}\tau_{b_0} = \sigma_{\tau_{b_0}(c_0)} = \sigma_{c_0},$$

i.e., τ_{b_0} commutes with σ_{c_0}. Therefore,

$$\tau_{a_0} = \tau_{\sigma_{c_0}(b_0)} = \sigma_{c_0}\tau_{b_0}\sigma_{c_0} = \tau_{b_0}$$

and hence by the structure of $6A$-algebra (see (4.0.17)),

$$x_5 = \tau_{a_0}(b_1) = \tau_{b_0}(b_1) = b_2.$$

Since (b_1, x_5, x_3, μ') is a normal $\mathcal{G}U_{3A}$ basis for $\mathcal{G}\{b_1, b_2\}$, we have

$$x_3 = \tau_{b_1}(x_5) = \tau_{b_1}(b_2) = b_0 \quad \text{and} \quad \mu' = \mu.$$

Note that μ and μ' are both determined by $b_0(= x_3), b_1, b_2(= x_5)$ using (4.0.3).

Recall that $(a_0, b_1, x_2, x_3, x_4, x_5, e, \mu')$ is the normal $\mathcal{G}U_{6A}$ basis for $\mathcal{G}\{a_0, b_1\}$. Thus, we have

$$e = \sigma_{a_0}(x_3) = \sigma_{a_0}(b_0) = c_0.$$

Finally, we will show that $\{a_1, a_2\} = \{x_2, x_4\}$. By (4.0.8), we have

$$\sigma_{a_0}(a_1 + a_2) = -\frac{3}{2^4}a_0 + \frac{a_1 + a_2}{2^2} + \frac{135}{2^7}\mu,$$

$$\sigma_{a_0}(x_2 + x_4) = -\frac{3}{2^4}a_0 + \frac{x_2 + x_4}{2^2} + \frac{135}{2^7}\mu'.$$

Note that $\mu = \mu'$ and hence

$$\begin{aligned}
& \langle a_1 + a_2, x_2 + x_4\rangle \\
= & \langle \sigma_{a_0}(a_1 + a_2), \sigma_{a_0}(x_2 + x_4)\rangle \\
= & \langle -\frac{3}{2^4}a_0 + \frac{a_1 + a_2}{2^2} + \frac{135}{2^7}\mu, -\frac{3}{2^4}a_0 + \frac{x_2 + x_4}{2^2} + \frac{135}{2^7}\mu\rangle \\
= & \frac{3}{2^4}\cdot\frac{3}{2^4}\cdot\frac{1}{2^2} + \frac{1}{2^4}\langle a_1 + a_2, x_2 + x_4\rangle + \frac{135^2}{2^{14}}\cdot\frac{2}{5} - 2\cdot\frac{3}{2^4}\cdot\frac{1}{2^2}(\frac{13}{2^{10}} + \frac{13}{2^{10}}) \\
& -2\cdot\frac{3}{2^4}\cdot\frac{135}{2^7}\cdot\frac{1}{2^4} + 2\cdot\frac{1}{2^2}\cdot\frac{135}{2^7}(\frac{1}{2^4} + \frac{1}{2^4}) \\
= & \frac{1}{2^4}\langle a_1 + a_2, x_2 + x_4\rangle + \frac{8070}{2^{14}},
\end{aligned}$$

which implies

$$\langle a_1 + a_2, x_2 + x_4\rangle = \frac{269}{2^9}.$$

On the other hand, we also have

$$\langle a_1 + a_2, a_1 + a_2\rangle = \frac{1}{2^2} + \frac{1}{2^2} + 2\cdot\frac{13}{2^{10}} = \frac{269}{2^9},$$

and similarly

$$\langle x_2 + x_4, x_2 + x_4\rangle = \frac{269}{2^9}.$$

Thus, by the Schwartz inequality, we get $a_1 + a_2 = x_2 + x_4$.

Taking inner product with a_1, we get

$$\langle a_1, x_2\rangle + \langle a_1, x_4\rangle = \langle a_1, x_2 + x_4\rangle = \langle a_1, a_1 + a_2\rangle = \frac{1}{2^2} + \frac{13}{2^{10}} = \frac{77}{2^{10}},$$

which implies $\max\{\langle a_1, x_2\rangle, \langle a_1, x_4\rangle\} \geq \frac{1}{2}\cdot\frac{77}{2^{10}} > \frac{1}{2^5}$. Then by Theorem 4.4, we have

$$(\langle a_1, x_2\rangle, \langle a_1, x_4\rangle) = (\frac{1}{2^2}, \frac{13}{2^{10}}) \text{ or } (\frac{13}{2^{10}}, \frac{1}{2^2}).$$

It implies $x_2 = a_1$ or $x_4 = a_1$. In either case, $\{x_2, x_4\} = \{a_1, a_2\}$. Therefore, $\mathcal{G} \subset \mathcal{G}\{a_0, b_1\}$ and thus $\mathcal{G} = \mathcal{G}\{a_0, b_1\} \cong \mathcal{G}U_{6A}$. □

6.3.3. Case3: $\mathcal{G}\{a_0, b_0\} \cong \mathcal{G}U_{2B}$. In this case, $a_0 \cdot b_0 = 0$ and $\langle a_0, b_0\rangle = 0$ by (4.0.2). Then, we have

$$\begin{aligned} 0 &= \langle a_0 \cdot b_0, \mu\rangle = \langle a_0, b_0 \cdot \mu\rangle \\ &= \langle a_0, \frac{2}{3^2}(2b_0 - b_1 - b_2) + \frac{5}{2^4}\mu\rangle \\ &= \frac{-2}{3^2}(\langle a_0, b_1\rangle + \langle a_0, b_2\rangle) + \frac{5}{2^4}\cdot\frac{1}{2^4}. \end{aligned}$$

Therefore we have

$$\langle a_0, b_1\rangle + \langle a_0, b_2\rangle = \frac{45}{2^9},$$

which implies $\max\{\langle a_0, b_1\rangle, \langle a_0, b_2\rangle\} \geq \frac{1}{2}\cdot\frac{45}{2^9} > \frac{1}{2^5}$. It means $a_0 = b_1$ or $a_0 = b_2$ since $\langle a_i, b_j\rangle \leq \frac{1}{2^5}$ if $a_i \neq b_j$. It is impossible since $\langle b_0, b_1\rangle = \langle b_0, b_2\rangle = \frac{13}{2^{10}}$ by our assumption.

6.3.4. Case4: $\mathcal{G}\{a_0, b_0\} \cong \mathcal{G}U_{3C}$. In this case, there is an Ising vector $c_0 \in \mathcal{G}$ such that (a_0, b_0, c_0) forms a normal $\mathcal{G}U_{3C}$ basis for $\mathcal{G}\{a_0, b_0\}$. Then we have

$$a_0 \cdot b_0 = \frac{1}{2^5}(a_0 + b_0 - c_0) \tag{6.3.8}$$

and

$$\langle a_0, b_0\rangle = \frac{1}{2^8} \tag{6.3.9}$$

by (4.0.10). Therefore,

$$\begin{aligned}\langle a_0 \cdot b_0, b_1\rangle &= \langle \frac{1}{2^5}(a_0 + b_0 - c_0), b_1\rangle \\ &= \frac{1}{2^5}(\langle a_0, b_1\rangle + \frac{13}{2^{10}} - \langle c_0, b_1\rangle).\end{aligned}$$

On the other hand,

$$\begin{aligned}\langle a_0 \cdot b_0, b_1\rangle &= \langle a_0, b_0 \cdot b_1\rangle \\ &= \langle a_0, \frac{1}{2^4}(2b_0 + 2b_1 + b_2) - \frac{135}{2^{10}}\mu\rangle \qquad \text{by (4.0.3)} \\ &= \frac{1}{2^4}\Big(2 \cdot \frac{1}{2^8} + 2\langle a_0, b_1\rangle + \langle a_0, b_2\rangle\Big) - \frac{135}{2^{10}} \cdot \frac{1}{2^4} \qquad \text{by (6.3.9)} \\ &= \frac{1}{2^4}\Big(2\langle a_0, b_1\rangle + \langle a_0, b_2\rangle\Big) - \frac{127}{2^{14}}.\end{aligned}$$

Combining these 2 equations we get

$$0 = \Big(3\langle a_0, b_1\rangle + 2\langle a_0, b_2\rangle + \langle c_0, b_1\rangle\Big) - \frac{267}{2^{10}}.$$

By Proposition 6.7, it is clear that $a_0 \neq b_1$, $a_0 \neq b_2$, $c_0 \neq b_1$. Thus, $\langle a_0, b_1\rangle, \langle a_0, b_2\rangle, \langle c_0, b_1\rangle \leq \frac{1}{2^5}$ and hence $\Big(3\langle a_0, b_1\rangle + 2\langle a_0, b_2\rangle + \langle c_0, b_1\rangle\Big) - \frac{267}{2^{10}} \leq 6 \cdot \frac{1}{2^5} - \frac{267}{2^{10}} = \frac{-75}{2^{10}} < 0$, which contradicts the above equation. So this case is impossible.

6.3.5. Case5: $\mathcal{G}\{a_0, b_0\} \cong \mathcal{G}U_{4A}$. In this case, there exist c_0, d_0, and u so that

$$\mathcal{G}\{a_0, b_0\} = \text{Span}\{a_0, b_0, c_0, d_0, u\}.$$

In addition, $\tau_{a_0}(b_0) = d_0$ and $\mathcal{G}\{b_0, d_0\} \cong \mathcal{G}U_{2B}$. Applying τ_{a_0} to the normal $\mathcal{G}U_{3A}$ basis (b_0, b_1, b_2, μ), we get another normal $\mathcal{G}U_{3A}$ basis $(d_0, \tau_{a_0}(b_1), \tau_{a_0}(b_2), \mu)$. Since $\mathcal{G}\{b_0, d_0\} \cong \mathcal{G}U_{2B}$, this case is also impossible by the analysis of $\mathcal{G}U_{2B}$ (see Chapter 6.3.3).

6.3.6. Case6: $\mathcal{G}\{a_0, b_0\} \cong \mathcal{G}U_{4B}$. In this case, there exist c_0, d_0, $e \in \mathcal{G}$ such that

$$\mathcal{G}\{a_0, b_0\} = \mathrm{Span}\{a_0, b_0, c_0, d_0, e\}$$

with $\mathcal{G}\{a_0, c_0\} = \mathrm{Span}\{a_0, c_0, e\} \cong \mathcal{G}U_{2A}$ and $\mathcal{G}\{b_0, d_0\} = \mathrm{Span}\{b_0, d_0, e\} \cong \mathcal{G}U_{2A}$ (see [**IPSS**, Table 3]). Moreover,

$$a_0 \cdot b_0 = \frac{1}{2^5}(a_0 + b_0 - c_0 - d_0 + e), \tag{6.3.10}$$

$$\langle a_0, b_0 \rangle = \frac{1}{2^8}, \tag{6.3.11}$$

and

$$\tau_{b_0}(a_0) = c_0$$

Applying τ_{b_0} to the normal $\mathcal{G}U_{3A}$ basis $\big(a_0, a_1, a_2, \mu\big)$, we get another normal $\mathcal{G}U_{3A}$ basis $\big(c_0, \tau_{b_0}(a_1),\ \tau_{b_0}(a_2), \mu\big)$. Then by Proposition 6.8, we have

$$\mathcal{G}\{a_0, a_1, a_2, c_0, \tau_{b_0}(a_1), \tau_{b_0}(a_2), \mu\} = \mathcal{G}\{c_0, a_1\} = \mathcal{G}\{a_0, \tau_{b_0}(a_1)\} \cong \mathcal{G}U_{6A}.$$

Set $x_0 := a_0,\ x_1 := \tau_{b_0}(a_1),\ x_3 := c_0,\ x_5 := \tau_{b_0}(a_2)$. Then there exists $\{x_2, x_4\} = \{a_1, a_2\}$ such that $(x_0, x_1, x_2, x_3, x_4,\ x_5, e, \mu)$ forms a normal $\mathcal{G}U_{6A}$ basis for $\mathcal{G}\{c_0, a_1\}$.

Similarly, set $y_0 := b_0,\ y_1 := \tau_{a_0}(b_1),\ y_3 := d_0,\ y_5 := \tau_{a_0}(b_2)$. There exists $\{y_2, y_4\} = \{b_1, b_2\}$, such that $(y_0, y_1, y_2, y_3, y_4, y_5, e, \mu)$ forms a normal $\mathcal{G}U_{6A}$ basis for $\mathcal{G}\{d_0, b_1\}$.

LEMMA 6.9. *For $i = 1, 2, 4, 5$, $\mathcal{G}\{x_0, y_i\} \cong \mathcal{G}\{x_3, y_i\} \cong \mathcal{G}U_{6A}$, and hence $\langle x_0, y_i \rangle = \langle x_3, y_i \rangle = \frac{5}{2^{10}}$. Similarly, $\langle x_i, y_0 \rangle = \langle x_i, y_3 \rangle = \frac{5}{2^{10}}$ for $i = 1, 2, 4, 5$.*

PROOF. Since (x_0, x_2, x_4, μ), (y_0, y_2, y_4, μ) are normal $\mathcal{G}U_{3A}$ bases, by Lemma 6.5, the order 3 element $\tau_{y_i}\tau_{y_0}$ commutes with $\tau_{y_0}\tau_{x_0}$ for $i = 2, 4$. Since $\mathcal{G}\{x_0, y_0\} \cong \mathcal{G}U_{4B}$, $\tau_{y_0}\tau_{x_0}$ has order 2 or 4. Hence $\tau_{y_i}\tau_{y_0} \cdot \tau_{y_0}\tau_{x_0}$ has order 6 or 12. Since $\tau_{y_i}\tau_{y_0} \cdot \tau_{y_0}\tau_{x_0} = \tau_{y_i}\tau_{x_0}$, by

6-transposition property (Theorem 4.4), $\tau_{y_i}\tau_{x_0}$ must have order ≤ 6 and hence has order 6 and $\mathcal{G}\{x_0, y_i\} \cong \mathcal{G}U_{6A}$ for $i = 2, 4$.

Since $(a_0, b_0, c_0, d_0) = (x_0, y_0, x_3, y_3)$ is a normal $\mathcal{G}U_{4B}$ basis, we have $\mathcal{G}\{x_0, y_3\} \cong \mathcal{G}U_{4B}$. Since (x_0, x_2, x_4, μ), (y_1, y_3, y_5, μ) form normal $\mathcal{G}U_{3A}$ bases, $\tau_{y_i}\tau_{y_3}$ commutes with $\tau_{y_3}\tau_{x_0}$ for $i = 1, 5$ and thus we also have $\mathcal{G}\{x_0, y_i\} \cong \mathcal{G}U_{6A}$ for $i = 1, 5$ by the same arguments as before. □

PROPOSITION 6.10. *It is impossible that* $\mathcal{G}\{a_0, b_0\} \cong \mathcal{G}U_{4B}$.

PROOF. By direct calculation, we have

$$\begin{aligned}
&\langle x_1 \cdot x_0, y_0\rangle \\
= \ &\langle \frac{1}{2^5}(x_0 + x_1 - x_2 - x_3 - x_4 - x_5 + e) + \frac{45}{2^{10}}\mu, y_0\rangle \quad \text{by (4.0.14)} \\
= \ &\frac{1}{2^5}\Big(\frac{1}{2^8} + \frac{5}{2^{10}} - \frac{5}{2^{10}} - \frac{1}{2^8} - \frac{5}{2^{10}} - \frac{5}{2^{10}} + \frac{1}{2^5}\Big) + \frac{45}{2^{10}} \cdot \frac{1}{2^4} \quad \text{by Lemma 6.9 and (6.3.11)} \\
= \ &\frac{7}{2^{11}},
\end{aligned}$$

and

$$\begin{aligned}
\langle x_1 \cdot x_0, y_0\rangle &= \langle x_1, x_0 \cdot y_0\rangle \\
&= \langle x_1, \frac{1}{2^5}(x_0 + y_0 - x_3 - y_3 + e)\rangle \qquad \text{by (6.3.10)} \\
&= \frac{1}{2^5}\Big(\frac{5}{2^{10}} + \frac{5}{2^{10}} - \frac{13}{2^{10}} - \frac{5}{2^{10}} + \frac{1}{2^5}\Big) \quad \text{by Lemma 6.9 and (4.0.16)} \\
&= \frac{3}{2^{12}},
\end{aligned}$$

which is a contradiction. So this case is impossible. □

6.3.7. Case7: $\mathcal{G}\{a_0, b_0\} \cong \mathcal{G}U_{5A}$. In this case, $\tau_{a_0}\tau_{b_0}$ has order 5.

PROPOSITION 6.11. *It is impossible that* $\mathcal{G}\{a_0, b_0\} \cong \mathcal{G}U_{5A}$.

Proof. By Lemma 6.5, the order 3 element $\tau_{a_1}\tau_{a_0}$ commutes with $\tau_{a_0}\tau_{b_0}$ and hence $\tau_{a_1}\tau_{a_0}\cdot\tau_{a_0}\tau_{b_0}$ has order 15. But $\tau_{a_1}\tau_{a_0}\cdot\tau_{a_0}\tau_{b_0} = \tau_{a_1}\tau_{b_0}$, which has order ≤ 6 by the 6-transposition property (Theorem 4.4). It is a contradiction. $\square$

6.3.8. Case8: $\mathcal{G}\{a_0, b_0\} \cong \mathcal{G}U_{6A}$. In this case, set $x_0 = a_0$, $x_1 = b_0$. Then there exist x_2, x_3, x_4, x_5, e, μ' such that the ordered set $(x_0, x_1, x_2, x_3, x_4, x_5, e, \mu')$ forms a normal $\mathcal{G}U_{6A}$ basis for $\mathcal{G}\{a_0, b_0\}$.

Proposition 6.12. *Suppose $\mathcal{G}\{a_0, b_0\} \cong \mathcal{G}U_{6A}$. Then $\mathcal{G} = \mathcal{G}\{a_0, b_0\}$.*

Proof. Since $\tau_{x_i}(x_j) = x_{2i-j}$ by (4.0.17) and μ is fixed by $\tau_{x_0} = \tau_{a_0}$ and $\tau_{x_1} = \tau_{b_0}$, we have

$$\langle x_4, \mu\rangle = \langle \tau_{x_0}x_2, \mu\rangle = \langle x_2, \mu\rangle = \langle \tau_{x_1}x_0, \mu\rangle = \langle x_0, \mu\rangle = \frac{1}{2^4}.$$

Similarly, we also have

$$\langle x_3, \mu\rangle = \langle \tau_{x_1}x_5, \mu\rangle = \langle x_5, \mu\rangle = \langle \tau_{x_0}x_1, \mu\rangle = \langle x_1, \mu\rangle = \frac{1}{2^4}.$$

Now let $h = \tau_{b_0}\tau_{a_0} = \tau_{x_1}\tau_{x_0}$. Then $\mathcal{G}\{h(b_0), h(b_1)\} \cong \mathcal{G}\{b_0, b_1\} \cong \mathcal{G}U_{3A}$ and the set $(h(b_0), h(b_1), h(b_2), h(\mu)) = (x_3, h(b_1), h(b_2), \mu)$ will form a normal $\mathcal{G}U_{3A}$ basis for $\mathcal{G}\{h(b_0), h(b_1)\}$. Note that $h(b_0) = h(x_1) = x_3$ and $h(\mu) = \tau_{b_0}\tau_{a_0}(\mu) = \mu$.

Since (a_0, x_3, e) forms a normal $\mathcal{G}U_{2A}$ basis for $\mathcal{G}\{a_0, x_3\}$, by Proposition 6.8, we have $\mathcal{G}\{a_0, a_1, x_3, h(b_1)\} = \mathcal{G}\{a_0, h(b_1)\} \cong \mathcal{G}U_{6A}$. Hence $\langle a_i, e\rangle = \frac{1}{2^5}$ for $i = 1, 2$ and $\langle e, \mu\rangle = 0$. Similarly we can also prove $\langle b_i, e\rangle = \frac{1}{2^5}$ for $i = 1, 2$.

Finally, we will show that $\{a_1, a_2\} = \{x_2, x_4\}$ and $\{b_1, b_2\} = \{x_3, x_5\}$.

By the structure of the 6A-algebra, we have

$$
\begin{aligned}
& \langle b_0 \cdot a_0, \mu\rangle \\
= & \langle \frac{1}{2^5}(x_0 + x_1 - x_2 - x_3 - x_4 - x_5 + e) + \frac{45}{2^{10}}\mu', \mu\rangle \qquad \text{by (4.0.14)} \\
= & \frac{1}{2^5}(\frac{1}{2^4} + \frac{1}{2^4} - \frac{1}{2^4} - \frac{1}{2^4} - \frac{1}{2^4} - \frac{1}{2^4} + 0) + \frac{45}{2^{10}}\langle \mu', \mu\rangle \qquad \text{by (4.0.6)} \\
= & -\frac{1}{2^8} + \frac{45}{2^{10}}\langle \mu', \mu\rangle,
\end{aligned}
$$

and

$$
\begin{aligned}
\langle b_0 \cdot a_0, \mu\rangle &= \langle b_0, a_0 \cdot \mu\rangle \\
&= \langle b_0, \frac{2}{3^2}(2a_0 - a_1 - a_2) + \frac{5}{2^4}\mu\rangle \qquad \text{by (4.0.4)} \\
&= \frac{2}{3^2}(2 \cdot \frac{5}{2^{10}} - \langle b_0, a_1\rangle - \langle b_0, a_2\rangle) + \frac{5}{2^4} \cdot \frac{1}{2^4} \qquad \text{by (4.0.16)} \\
&= \frac{50}{2^8 \cdot 3^2} - \frac{2}{3^2}(\langle b_0, a_1\rangle + \langle b_0, a_2\rangle),
\end{aligned}
$$

which implies

(6.3.12) $$\langle \mu', \mu\rangle = \frac{2^2}{3^4 \cdot 5}\big(59 - 2^9(\langle b_0, a_1\rangle + \langle b_0, a_2\rangle)\big).$$

Since $\mathcal{G}\{x_0, x_2\} \cong \mathcal{G}U_{3A}$, we also have

$$
\begin{aligned}
\langle x_2 \cdot x_0, \mu\rangle &= \langle \frac{1}{2^4}(2x_0 + 2x_2 + x_4) - \frac{135}{2^{10}}\mu', \mu\rangle \qquad \text{by (4.0.3)} \\
&= \frac{1}{2^4}(2 \cdot \frac{1}{2^4} + 2 \cdot \frac{1}{2^4} + \frac{1}{2^4}) - \frac{135}{2^{10}}\langle \mu', \mu\rangle \\
&= \frac{5}{2^8} - \frac{135}{2^{10}}\langle \mu', \mu\rangle,
\end{aligned}
$$

and

$$\begin{aligned}\langle x_2 \cdot x_0, \mu\rangle &= \langle x_2, x_0 \cdot \mu\rangle = \langle x_2, a_0 \cdot \mu\rangle \\ &= \langle x_2, \frac{2}{3^2}(2a_0 - a_1 - a_2) + \frac{5}{2^4}\mu\rangle \\ &= \frac{2}{3^2}(2 \cdot \frac{13}{2^{10}} - \langle x_2, a_1\rangle - \langle x_2, a_2\rangle) + \frac{5}{2^4} \cdot \frac{1}{2^4} \\ &= \frac{58}{2^8 \cdot 3^2} - \frac{2}{3^2}(\langle x_2, a_1\rangle + \langle x_2, a_2\rangle),\end{aligned}$$

which implies

$$\langle \mu', \mu\rangle = \frac{2^2}{3^5 \cdot 5}\big(-13 + 2^9(\langle x_2, a_1\rangle + \langle x_2, a_2\rangle)\big). \tag{6.3.13}$$

From (6.3.12) and (6.3.13), we get

$$3\langle b_0, a_1\rangle + 3\langle b_0, a_2\rangle + \langle x_2, a_1\rangle + \langle x_2, a_2\rangle = \frac{95}{2^8},$$

which implies

$$\max\{\langle b_0, a_1\rangle, \langle b_0, a_2\rangle, \langle x_2, a_1\rangle, \langle x_2, a_2\rangle\} \geq \frac{95}{2^8(3+3+1+1)} > \frac{1}{2^5}. \tag{6.3.14}$$

By Proposition 6.7, $a_i \neq b_j$ for any $i, j \in \{0, 1, 2\}$ and thus we must have $x_2 = a_1$ or $x_2 = a_2$. A similar argument also shows that $x_3 = b_1$ or b_2. Therefore, $\mathcal{G} = \mathcal{G}\{a_0, b_0\} \cong \mathcal{G}U_{6A}$. □

6.3.9. Case9: $\mathcal{G}\{a_0, b_0\} \cong \mathcal{G}U_{3A}$. By assumption, there exists c_0 and μ_0 such that (a_0, b_0, c_0, μ_0) forms a normal $\mathcal{G}U_{3A}$ basis.

Lemma 6.13. *Let (a_0, a_1, a_2, μ) and (b_0, b_1, b_2, μ) be normal $\mathcal{G}U_{3A}$ bases and $\mathcal{G}\{a_0, b_0\} \cong \mathcal{G}U_{3A}$. Then either*

(1) *$\{a_0, a_1, a_2\} = \{b_0, b_1, b_2\}$ and $\mathcal{G} \cong \mathcal{G}U_{3A}$; or*

(2) *$\mathcal{G}\{a_i, b_j\} \cong \mathcal{G}U_{3A}$ for $i, j \in \mathbb{Z}_3$.*

PROOF. By Lemma 6.5, for $i = 1, 2$, the order 3 element $\tau_{a_i}\tau_{a_0}$ commutes with $\tau_{a_0}\tau_{b_0}$, which has order 3 by assumption. Hence $\tau_{a_i}\tau_{a_0} \cdot \tau_{a_0}\tau_{b_0} = \tau_{a_i}\tau_{b_0}$ has order 1 or 3 for $i = 1, 2$.

Case 1. $\tau_{a_i}\tau_{b_0}$ is of order 1. Then $\tau_{a_i}\tau_{a_0} = (\tau_{a_0}\tau_{b_0})^{-1}$ and we have

$$a_j = \tau_{a_i}\tau_{a_0}a_0 = \tau_{b_0}\tau_{a_0}a_0 = c_0,$$

where $\{0, i, j\} = \{0, 1, 2\}$. Thus, by Proposition 6.7, we have $b_0 \in \{a_0, a_1, a_2\}$ and $\{a_0, a_1, a_2\} = \{b_0, b_1, b_2\}$.

Case 2. $\tau_{a_i}\tau_{b_0}$ has order 3. Then $\mathcal{G}\{a_i, b_0\} \cong \mathcal{G}U_{3A}$, $\mathcal{G}U_{3C}$ or $\mathcal{G}U_{6A}$.

By the discussion in Chapter 6.3.4, $\mathcal{G}\{a_i, b_0\} \cong \mathcal{G}U_{3C}$ is impossible.

If $\mathcal{G}\{a_i, b_0\} \cong \mathcal{G}U_{6A}$, then by Proposition 6.12, $\langle a_0, b_0\rangle = \frac{1}{32}$ or $\frac{5}{2^{10}}$, which is again impossible since $\mathcal{G}\{a_0, b_0\} \cong \mathcal{G}U_{3A}$. Therefore, $\mathcal{G}\{a_i, b_0\} \cong \mathcal{G}U_{3A}$ is the only possible case. Similarly, we also have $\mathcal{G}\{a_i, b_j\} \cong \mathcal{G}U_{3A}$ for any $i, j = 0, 1, 2$. □

From now on, we assume $\{a_0, a_1, a_2\} \neq \{b_0, b_1, b_2\}$, which implies $\mathcal{G}\{a_i, b_j\} \cong \mathcal{G}U_{3A}$ for all $i \neq j$.

NOTATION 6.14. *Let $g := \tau_{a_0}\tau_{a_1} = \tau_{a_2}\tau_{a_0}$ and $h := \tau_{a_0}\tau_{b_0}$. Then both g and h are of order 3 and g commutes with h by Lemma 6.5. Moreover, we have*

$$\tau_{a_2}\tau_{b_0} = \tau_{a_2}\tau_{a_0} \cdot \tau_{a_0}\tau_{b_0} = gh,$$

$$\tau_{a_1}\tau_{b_0} = \tau_{a_1}\tau_{a_0} \cdot \tau_{a_0}\tau_{b_0} = g^2h.$$

For $i, j = 0, 1, 2$, denote

$$x_{i,j} = h^i g^j(a_0).$$

Note that $x_{0,0} = a_0$, $x_{0,1} = g(a_0) = a_1$, $x_{0,2} = g^2(a_0) = a_2$, and $x_{1,0} = h(a_0) = b_0$. By definition, it is also easy to see that

$$h^k g^\ell(x_{i,j}) = x_{i+k,\,j+\ell}, \qquad \text{for } i, j, k, \ell \in \mathbb{Z}_3.$$

Notation 6.15. *For any* $(i,j) \neq (0,0)$*, denote*

$$\mathcal{G}_{i,j,0} := \mathcal{G}\{x_{0,0}, x_{i,j}\} \cong \mathcal{G}U_{3A}.$$

Then there exists a conformal vector $\mu_{i,j,0}$ *of central charge* $4/5$ *such that* $(x_{0,0}, x_{i,j}, x_{2i,2j}, \mu_{i,j,0})$ *forms a normal* $\mathcal{G}U_{3A}$ *basis of* $\mathcal{G}_{i,j,0}$*. For* $k = 1, 2$*, we denote*

$$\mathcal{G}_{0,1,k} := h^k(\mathcal{G}_{0,1,0}) = h^k(\mathcal{G}_{0,2,0}).$$

Then $\mathcal{G}_{0,1,k} \cong \mathcal{G}U_{3A}$ *and there is a conformal vector* $\mu_{0,1,k}$ *of central charge* $4/5$ *such that* $(x_{k,0}, x_{k,1}, x_{k,2}, \mu_{0,1,k})$ *forms a normal basis for* $\mathcal{G}_{0,1,k}$*.*

Remark 6.16. By our assumption, we have $\mu_{(0,1,0)} = \mu_{(0,1,1)} = \mu_{(0,1,2)} = \mu$. We use $\mu_{0,1}$ to denote $\mu_{(0,1,0)} = \mu_{(0,1,1)} = \mu_{(0,1,2)}$. Note that $\mu_{0,1}$ is fixed by $\tau_{x_{i,j}}$ for all i, j.

Notation 6.17. *For* $(i,j) \neq (0,0), (0,1)$ *and* $(0,2)$*, we denote*

$$\mathcal{G}_{i,j,k} = g^k(\mathcal{G}_{i,j,0}).$$

Then, $\mathcal{G}_{i,j,k} \cong \mathcal{G}U_{3A}$ *for any* $k = 0, 1, 2$*. Let* $\mu_{i,j,k}$ *be the conformal vector of central charge* $4/5$ *such that* $(x_{0,k}, x_{i,j+k}, x_{2i,2j+k}, \mu_{i,j,k})$ *forms a normal* $\mathcal{G}U_{3A}$ *basis for* $\mathcal{G}_{i,j,k}$*. Note that* $\mu_{i,j,k} = \mu_{2i,2j,k}$ *and* $g^\ell(\mu_{i,j,k}) = \mu_{i,j,k+\ell}$ *for any* $i \neq 0$*.*

We will show $\mu_{1,i,j} = \mu_{1,i,k}$ for all i, j, k (Proposition 6.27). This turns out to be the most complicated part of the proof.

Lemma 6.18. *For any* $n, i, k, \ell \in \mathbb{Z}_3$*, we have*

$$\tau_{x_{n,n\ell+k}}(\mu_{1,\ell,i}) = \mu_{1,\ell,-k-i}. \tag{6.3.15}$$

Proof. By definition, we have by (6.2.2),

$$\tau_{x_{i,j}}(x_{k,\ell}) = h^i g^j \tau_{a_0} g^{-j} h^{-i} h^k g^\ell (a_0) = h^{-k+2i} g^{-\ell+2j} \tau_{a_0}(a_0) = x_{-i-k,-j-\ell}.$$

Thus, $\tau_{x_{n,n\ell+k}}$ maps the normal $\mathcal{G}U_{3A}$ basis $(x_{0,i}, x_{1,i+\ell}, x_{2,i+2\ell}, \mu_{1,\ell,i})$ to

$$(x_{-n,-n\ell-k-i}, x_{-n-1,-n\ell-k-i-\ell}, x_{-n-2,-n\ell-k-i-2\ell}, \tau_{x_{n,n\ell+k}}(\mu_{1,\ell,i})).$$

Then we have

$$\begin{aligned} & \{x_{-n,-n\ell-k-i}, x_{-n-1,-n\ell-k-i-\ell}, x_{-n-2,-n\ell-k-i-2\ell}\} \\ = & \{x_{0,-k-i}, x_{-1,-k-i-\ell}, x_{-2,-k-i-2\ell}\} \\ = & \{x_{0,-k-i}, x_{2,-k-i+2\ell}, x_{1,-k-i+\ell}\}. \end{aligned}$$

Since $(x_{0,-k-i}, x_{1,-k-i+\ell}, x_{2,-k-i+2\ell}, \mu_{1,\ell,-k-i})$ forms a normal $\mathcal{G}U_{3A}$ basis, we have that $\tau_{x_{n,n\ell+k}}(\mu_{1,\ell,i}) = \mu_{1,\ell,-k-i}$. □

Lemma 6.19. *For any $i, j \in \mathbb{Z}_3$, $y \in \{\mu_{0,1}, \mu_{1,0,k}, \mu_{1,1,k}, \mu_{1,2,k} | \, k = 0, 1, 2\}$, we have*

$$\langle x_{i,j}, y \rangle = \frac{1}{2^4} \tag{6.3.16}$$

and

$$\langle \mu_{1,i,j}, \mu_{1,k,\ell} \rangle = 0, \quad \langle \mu_{0,1}, \mu_{1,i,j} \rangle = 0 \tag{6.3.17}$$

for all i, j, and for $k \neq i$.

Proof. Computing

$$\begin{aligned} \langle x_{0,0}, x_{0,1} \cdot x_{1,0} \rangle &= \langle x_{0,0}, \frac{1}{2^4}(2x_{0,1} + 2x_{1,0} + x_{2,2}) - \frac{135}{2^{10}} \mu_{1,2,1} \rangle \qquad \text{by (4.0.3)} \\ &= \frac{1}{2^4}(2 \cdot \frac{13}{2^{10}} + 2 \cdot \frac{13}{2^{10}} + \frac{13}{2^{10}}) - \frac{135}{2^{10}} \langle x_{0,0}, \mu_{1,2,1} \rangle \\ &= \frac{65}{2^{14}} - \frac{135}{2^{10}} \langle x_{0,0}, \mu_{1,2,1} \rangle \end{aligned}$$

together with

$$\begin{aligned}
\langle x_{0,0}, x_{0,1}\cdot x_{1,0}\rangle &= \langle x_{0,0}\cdot x_{0,1}, x_{1,0}\rangle \\
&= \langle \frac{1}{2^4}(2x_{0,1}+2x_{1,0}+x_{2,2}) - \frac{135}{2^{10}}\mu_{0,1}, x_{1,0}\rangle \\
&= \frac{1}{2^4}(2\cdot\frac{13}{2^{10}} + 2\cdot\frac{13}{2^{10}} + \frac{13}{2^{10}}) - \frac{135}{2^{10}}\cdot\frac{1}{2^4} \\
&= \frac{65}{2^{14}} - \frac{135}{2^{10}}\cdot\frac{1}{2^4},
\end{aligned}$$

we can get $\langle x_{0,0}, \mu_{1,2,1}\rangle = \frac{1}{2^4}$. Similarly, we can get (6.3.16).

Computing

$$\begin{aligned}
\langle \mu_{1,0,1}, x_{0,1}\cdot x_{1,0}\rangle &= \langle \mu_{1,0,1}, \frac{1}{2^4}(2x_{0,1}+2x_{1,0}+x_{2,2}) - \frac{135}{2^{10}}\mu_{1,2,1}\rangle \\
&= \frac{1}{2^4}(2\cdot\frac{1}{2^4} + 2\cdot\frac{1}{2^4} + \frac{1}{2^4}) - \frac{135}{2^{10}}\langle \mu_{1,0,1}, \mu_{1,2,1}\rangle \qquad \text{by (6.3.16)} \\
&= \frac{5}{2^8} - \frac{135}{2^{10}}\langle \mu_{1,0,1}, \mu_{1,2,1}\rangle
\end{aligned}$$

together with

$$\begin{aligned}
\langle \mu_{1,0,1}, x_{0,1}\cdot x_{1,0}\rangle &= \langle \mu_{1,0,1}\cdot x_{0,1}, x_{1,0}\rangle \\
&= \langle \frac{2}{3^2}(2x_{0,1} - x_{2,1} - x_{1,1}) + \frac{5}{2^4}\mu_{1,0,1}, x_{1,0}\rangle \\
&= \frac{2}{3^2}(2\cdot\frac{13}{2^{10}} - \frac{13}{2^{10}} - \frac{13}{2^{10}}) + \frac{5}{2^4}\cdot\frac{1}{2^4} \qquad \text{by (6.3.16)} \\
&= \frac{5}{2^8},
\end{aligned}$$

we can get

$$\langle \mu_{1,0,1}, \mu_{1,2,1}\rangle = 0.$$

Similar argument gives (6.3.17). □

LEMMA 6.20. *We have*

$$\mu_{1,i,j}\cdot\mu_{1,k,\ell} = 0 \tag{6.3.18}$$

for $i \neq k$, *and*

$$\mu_{0,1} \cdot \mu_{1,i,j} = 0 \tag{6.3.19}$$

for $i \in \mathbb{Z}_3$.

PROOF. By Theorem 2.27, Norton inequality, and (6.3.17) we have

$$\begin{aligned}
|\mu_{1,i,j} \cdot \mu_{1,k,\ell}|^2 &= \langle \mu_{1,i,j} \cdot \mu_{1,k,\ell}, \mu_{1,i,j} \cdot \mu_{1,k,\ell} \rangle \\
&\leq \langle \mu_{1,i,j} \cdot \mu_{1,i,j}, \mu_{1,k,\ell} \cdot \mu_{1,k,\ell} \rangle \\
&= \langle 2\mu_{1,i,j}, 2\mu_{1,k,\ell} \rangle \qquad \text{by (4.0.5)} \\
&= 0.
\end{aligned}$$

Since the inner product is positive definite by Assumption 1, we have (6.3.18). Similarly, we can get (6.3.19). □

LEMMA 6.21. *For* $x \in \{x_{i,j} \,|\, i,j\}$, $\mu' \in \{\mu_{0,1}, \mu_{1,i,j} \,|\, i,j\}$, *we have*

$$x \cdot \mu' = \frac{1}{2}x + \frac{5}{2^5}\mu' + \frac{3}{2^5}\tau_x(\mu') - \frac{1}{2^3}\sigma_x(\mu' + \tau_x(\mu')). \tag{6.3.20}$$

PROOF. By Lemma 2.24 and (6.3.16) we have

$$\begin{aligned}
x \cdot \mu' &= 8\langle x, \mu' \rangle x + \frac{1}{2^2}\left(\frac{1}{2}(\mu' + \tau_x(\mu')) - \sigma_x\big(\frac{1}{2}(\mu' + \tau_x(\mu'))\big)\right) + \frac{1}{2^5}\big(\mu' - \tau_x(\mu')\big) \\
&= 8 \cdot \frac{1}{2^4} \cdot x + \frac{1}{2^2}\big(\frac{1}{2}(\mu' + \tau_x(\mu')) - \frac{1}{2}\sigma_x(\mu' + \tau_x(\mu'))\big) + \frac{1}{2^5}(\mu' - \tau_x(\mu')) \\
&= \frac{1}{2}x + \frac{5}{2^5}\mu' + \frac{3}{2^5}\tau_x(\mu') - \frac{1}{2^3}\sigma_x(\mu' + \tau_x(\mu')).
\end{aligned}$$

□

LEMMA 6.22. *For* $i \in \{0,1,2\}$, *we have*

$$\langle \mu_{1,i,0}, \mu_{1,i,2} \rangle = \langle \mu_{1,i,1}, \mu_{1,i,2} \rangle = \langle \mu_{1,i,0}, \mu_{1,i,1} \rangle. \tag{6.3.21}$$

PROOF. Since $g \in \mathrm{Aut}(\mathcal{G})$ preserve the inner product, we have

$$\langle \mu_{1,i,0}, \mu_{1,i,1} \rangle = \langle g^j(\mu_{1,i,0}), g^j(\mu_{1,i,1}) \rangle = \langle \mu_{1,i,j}, \mu_{1,i,1+j} \rangle$$

for any $j = 0, 1, 2$. □

LEMMA 6.23. *For $x = x_{k,\ell}$, $\mu' = \mu_{1,i,j}$, $\mu'' = \tau_x(\mu')$, we have*

$$\langle \sigma_x(\mu' + \mu''), \mu' \rangle = \frac{-1}{2^2} + \frac{3}{2^2} \langle \mu', \mu'' \rangle. \tag{6.3.22}$$

PROOF. By (6.3.20), (6.3.16), and (6.3.17), we have

$$\begin{aligned}
& \langle x \cdot \mu', \mu' \rangle \\
= \; & \langle \frac{1}{2} x + \frac{5}{2^5} \mu' + \frac{3}{2^5} \mu'' - \frac{1}{2^3} \sigma_x(\mu' + \mu''), \mu' \rangle \\
= \; & \frac{1}{2^5} + \frac{1}{2^4} + \frac{3}{2^5} \langle \mu', \mu'' \rangle - \frac{1}{2^3} \langle \sigma_x(\mu' + \mu''), \mu' \rangle \\
= \; & \frac{3}{2^5} + \frac{3}{2^5} \langle \mu', \mu'' \rangle - \frac{1}{2^3} \langle \sigma_x(\mu' + \mu''), \mu' \rangle.
\end{aligned}$$

By (4.0.5), we also have

$$\langle x \cdot \mu', \mu' \rangle = \langle x, \mu' \cdot \mu' \rangle = 2 \langle x, \mu' \rangle = \frac{1}{2^3}.$$

Hence we get

$$\langle \sigma_x(\mu' + \mu''), \mu' \rangle = \frac{-1}{2^2} + \frac{3}{2^2} \langle \mu', \mu'' \rangle$$

as desired. □

LEMMA 6.24. *Let $\mu' = \mu_{i,j,k}$ and $\mu'' = \mu_{i',j',k'}$. If $(i,j) \neq (i',j')$ or $(2i', 2j')$, then we have*

$$\langle \sigma_x(\mu' + \tau_x(\mu')), \mu'' \rangle = \frac{1}{2^2}. \tag{6.3.23}$$

for any $x = x_{k,\ell}$.

Proof. By Lemma 6.20, we have $\mu' \cdot \mu'' = 0$ and $\langle \mu', \mu'' \rangle = \langle \tau_x(\mu'), \mu'' \rangle = 0$. Hence

$$
\begin{aligned}
0 &= \langle x, \mu' \cdot \mu'' \rangle \\
&= \langle x \cdot \mu', \mu'' \rangle \\
&= \langle \frac{1}{2}x + \frac{5}{2^5}\mu' + \frac{3}{2^5}\tau_x(\mu') - \frac{1}{2^3}\sigma_x(\mu' + \tau_x(\mu')), \mu'' \rangle \quad \text{by (6.3.20)} \\
&= \frac{1}{2} \cdot \frac{1}{2^4} + \frac{5}{2^5} \cdot 0 + \frac{3}{2^5} \cdot 0 - \frac{1}{2^3}\langle \sigma_x(\mu' + \tau_x(\mu')), \mu'' \rangle,
\end{aligned}
$$

which implies (6.3.23). $\square$

Lemma 6.25. *We have*

$$
\begin{aligned}
& 6075\mu_{0,1} \cdot \mu_{1,1,1} \\
= \; & 64x_{0,1} - 656(x_{0,0} + x_{0,2}) - 576(x_{1,2} + x_{2,0}) + 384(x_{1,0} + x_{1,1} + x_{2,1} + x_{2,2}) \\
& +810\mu_{0,1} + 1260\mu_{1,1,1} - 135(\mu_{1,1,0} + \mu_{1,1,2}) + 360(\mu_{1,0,1} + \mu_{1,2,1}) \\
& +45(\mu_{1,0,0} + \mu_{1,0,2} + \mu_{1,2,0} + \mu_{1,2,2}) - 720(\sigma_{x_{0,1}}(\mu_{1,0,0} + \mu_{1,0,2} + \mu_{1,2,0} + \mu_{1,2,2}) \\
& +180\big(\sigma_{x_{0,0}}(\mu_{1,1,1} + \mu_{1,1,2}) + \sigma_{x_{0,2}}(\mu_{1,1,0} + \mu_{1,1,1})\big) \\
= \; & 0.
\end{aligned}
$$

Proof. We will expand both sides of the equality

$$
\sigma_{x_{0,1}}\big((x_{0,2} + x_{0,0}) \cdot (x_{1,2} + x_{2,0})\big) = \sigma_{x_{0,1}}(x_{0,2} + x_{0,0}) \cdot \sigma_{x_{0,1}}(x_{1,2} + x_{2,0}).
$$

First we compute

$$
\begin{aligned}
& \sigma_{x_{0,1}}\big((x_{0,2}+x_{0,0})\cdot(x_{1,2}+x_{2,0})\big) \\
= \;& \sigma_{x_{0,1}}\Big(\frac{1}{2^4}(2x_{0,2}+2x_{1,2}+x_{2,2})+\frac{1}{2^4}(2x_{0,2}+2x_{2,0}+x_{1,1})+\frac{1}{2^4}(2x_{0,0}+2x_{1,2}+x_{2,1}) \\
& +\frac{1}{2^4}(2x_{0,0}+2x_{2,0}+x_{1,0})-\frac{135}{2^{10}}(\mu_{1,0,2}+\mu_{1,2,2}+\mu_{1,2,0}+\mu_{1,0,0})\Big) \\
= \;& \frac{1}{2^4}x_{0,0}-\frac{15}{2^7}x_{0,1}+\frac{1}{2^4}x_{0,2}+\frac{1}{2^6}x_{1,0}+\frac{1}{2^6}x_{1,1}+\frac{1}{2^4}x_{1,2}+\frac{1}{2^4}x_{2,0}+\frac{1}{2^6}x_{2,1}+\frac{1}{2^6}x_{2,2} \\
& +\frac{135}{2^9}\mu_{0,1}+\frac{135}{2^9}\mu_{1,1,1}+\frac{135}{2^{11}}\mu_{1,0,1}+\frac{135}{2^{11}}\mu_{1,2,1} \\
& -\frac{135}{2^{10}}\sigma_{x_{0,1}}(\mu_{1,0,0}+\mu_{1,0,2})-\frac{135}{2^{10}}\sigma_{x_{0,1}}(\mu_{1,2,0}+\mu_{1,2,2}) \quad \text{by (4.0.8).}
\end{aligned} \tag{6.3.24}
$$

By (4.0.8), (6.3.20), and (6.3.15), we also have

$$
\begin{aligned}
& \sigma_{x_{0,1}}(x_{0,2}+x_{0,0})\cdot\sigma_{x_{0,1}}(x_{1,2}+x_{2,0}) \\
= \;& (\frac{-3}{2^4}x_{0,1}+\frac{1}{2^2}x_{0,2}+\frac{1}{2^2}x_{0,0}+\frac{135}{2^7}\mu_{0,1})\cdot(\frac{-3}{2^4}x_{0,1}+\frac{1}{2^2}x_{1,2}+\frac{1}{2^2}x_{2,0}+\frac{135}{2^7}\mu_{1,1,1}) \\
= \;& \frac{187}{2^{10}}x_{0,0}-\frac{33}{2^8}x_{0,1}+\frac{187}{2^{10}}x_{0,2}-\frac{7}{2^7}x_{1,0}-\frac{7}{2^7}x_{1,1}+\frac{43}{2^8}x_{1,2}+\frac{43}{2^8}x_{2,0}-\frac{7}{2^7}x_{2,1}-\frac{7}{2^7}x_{2,2} \\
& +\frac{945}{2^{13}}\mu_{0,1}-\frac{135}{2^{14}}(\mu_{1,0,0}+\mu_{1,0,2}+\mu_{1,2,0}+\mu_{1,2,2})+\frac{135}{2^{12}}\mu_{1,1,1}+\frac{405}{2^{14}}(\mu_{1,1,0}+\mu_{1,1,2}) \\
& -\frac{135}{2^{12}}\big(\sigma_{x_{0,0}}(\mu_{1,1,1}+\mu_{1,1,2})+\sigma_{x_{0,2}}(\mu_{1,1,0}+\mu_{1,1,1})\big)+\frac{18225}{2^{14}}\mu_{0,1}\cdot\mu_{1,1,1}.
\end{aligned} \tag{6.3.25}
$$

Hence we have by (6.3.19), (6.3.24), (6.3.25),

$$
\begin{aligned}
0 \;=\;& 6075\mu_{0,1}\cdot\mu_{1,1,1} \\
=\;& 64x_{0,1}-656(x_{0,0}+x_{0,2})-576(x_{1,2}+x_{2,0})+384(x_{1,0}+x_{1,1}+x_{2,1}+x_{2,2}) \\
& +810\mu_{0,1}+1260\mu_{1,1,1}-135(\mu_{1,1,0}+\mu_{1,1,2})+360(\mu_{1,0,1}+\mu_{1,2,1}) \\
& +45(\mu_{1,0,0}+\mu_{1,0,2}+\mu_{1,2,0}+\mu_{1,2,2})-720(\sigma_{x_{0,1}}(\mu_{1,0,0}+\mu_{1,0,2}+\mu_{1,2,0}+\mu_{1,2,2}) \\
& +180\big(\sigma_{x_{0,0}}(\mu_{1,1,1}+\mu_{1,1,2})+\sigma_{x_{0,2}}(\mu_{1,1,0}+\mu_{1,1,1})\big),
\end{aligned}
$$

as desired. □

LEMMA 6.26. *For $i, k, \ell \in \{0, 1, 2\}$, we have*

$$\langle \mu_{1,i,k}, \mu_{1,i,\ell} \rangle = \frac{2}{5}. \tag{6.3.26}$$

Hence, $\mu_{1,i,k} = \mu_{1,i,\ell}$ for any i, k, ℓ.

PROOF. By Lemma 6.25 and (6.3.21), (6.3.22), and (6.3.23), we have

$$\begin{aligned}
0 &= \langle 6075\mu_{0,1} \cdot \mu_{1,1,1}, \mu_{1,0,0} \rangle \\
&= 64 \cdot \frac{1}{2^4} - 656(\frac{1}{2^4} + \frac{1}{2^4}) - 576(\frac{1}{2^4} + \frac{1}{2^4}) + 384(\frac{1}{2^4} + \frac{1}{2^4} + \frac{1}{2^4} + \frac{1}{2^4}) \\
&\quad +810 \cdot 0 + 1260 \cdot 0 - 135(0 + 0) + 360(\langle \mu_{1,0,0}, \mu_{1,0,1} \rangle + 0) \\
&\quad +45(\frac{2}{5} + \langle \mu_{1,0,0}, \mu_{1,0,1} \rangle + 0 + 0) - 720(-\frac{1}{2^2} + \frac{3}{2^2}\langle \mu_{1,0,0}, \mu_{1,0,1} \rangle + \frac{1}{2^2}) \\
&\quad +180(\frac{1}{2^2} + \frac{1}{2^2}) \\
&= 54 - 135\langle \mu_{1,0,0}, \mu_{1,0,1} \rangle.
\end{aligned}$$

which implies $\langle \mu_{1,0,0}, \mu_{1,0,1} \rangle = \frac{2}{5}$. Similarly, one can prove $\langle \mu_{1,i,k}, \mu_{1,i,\ell} \rangle = \frac{2}{5}$, also. □

NOTATION 6.27. *Denote*

$$\mu_{1,0} := \mu_{1,0,0} = \mu_{1,0,1} = \mu_{1,0,2},$$

$$\mu_{1,1} := \mu_{1,1,0} = \mu_{1,1,1} = \mu_{1,1,2},$$

$$\mu_{1,2} := \mu_{1,2,0} = \mu_{1,2,1} = \mu_{1,2,2}.$$

PROPOSITION 6.28. *For any $(i, j) \neq (i', j')$,we have*

$$\mu_{i,j} \cdot \mu_{i',j'} = 0. \tag{6.3.27}$$

Moreover,

$$\mu_{0,1} + \mu_{1,0} + \mu_{1,1} + \mu_{1,2} = \frac{32}{45}(x_{0,0} + x_{0,1} + x_{0,2} + x_{1,0} + x_{1,1} + x_{1,2} + x_{2,0} + x_{2,1} + x_{2,2}). \tag{6.3.28}$$

Therefore, the dimension of $\mathcal{G}$ is 12.

Proof. The first assertion follows from (6.3.17) and Lemma 6.26.

To prove (6.3.28), let

$$\tilde{\mu} = \mu_{0,1} + \mu_{1,0} + \mu_{1,1} + \mu_{1,2},$$

$$\tilde{x} = \frac{32}{45}(x_{0,0} + x_{0,1} + x_{0,2} + x_{1,0} + x_{1,1} + x_{1,2} + x_{2,0} + x_{2,1} + x_{2,2}).$$

Then by Lemmas 6.19, 6.25, and $\langle \mu_{i,j}, \mu_{i',j'} \rangle = 0$ for $(i,j) \neq (i'j')$, we have

$$\langle \tilde{\mu} - \tilde{x}, \tilde{\mu} - \tilde{x} \rangle = 0$$

and thus $\tilde{\mu} = \tilde{x}$ as desired.

To check the dimension of $\mathcal{G}$, for $\{a_1, a_2, \cdots, a_{12}\} = \{x_{i,j}, \mu_{0,1}, \mu_{1,0}, \mu_{1,1} \,|\, i, j \in \mathbb{Z}_3\}$, we can get $\det(\langle a_i, a_j \rangle) = \frac{3^{42}}{2^{86} \cdot 5^2} \neq 0$ by computer. Hence the dimension of $\mathcal{G}$ is 12. □

The structure of $\mathcal{G}V_{F(3A)}$ is summarized in Figure 1.

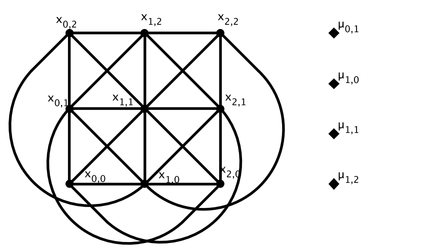


Figure 1. Configuration for $\mathcal{G}V_{\mathbb{B}(4B)}$

To summarize, we have proved the theorem.

Theorem 6.29. *Let (a_0, a_1, a_2, μ) and (b_0, b_1, b_2, μ) be normal $\mathcal{G}U_{3A}$ bases of $\mathcal{G}U$ and $\mathcal{G}U'$, respectively. Let $\mathcal{G}$ be the sub-Griess algebra generated by $\{a_0, a_1, a_2, b_0, b_1, b_2, \mu\}$. Then, it is impossible that $\mathcal{G}\{a_0, b_0\} \cong \mathcal{G}U_{2B}$, $\mathcal{G}U_{3C}$, $\mathcal{G}U_{4A}$, $\mathcal{G}U_{4B}$ and $\mathcal{G}U_{5A}$.*

(1) *If $\mathcal{G}\{a_0, b_0\} \cong \mathcal{G}U_{1A}$, then $\mathcal{G} \cong \mathcal{G}V_{F(1A)} \cong \mathcal{G}U_{3A}$ and $\{a_0, a_1, a_2\} = \{b_0, b_1, b_2\}$.*

(2) *If $\mathcal{G}\{a_0, b_0\} \cong \mathcal{G}U_{2A}$ or $\mathcal{G}U_{6A}$, then $\mathcal{G} \cong \mathcal{G}V_{F(2A)} \cong \mathcal{G}U_{6A}$.*

(3) *If $\mathcal{G}\{a_0, b_0\} \cong \mathcal{G}U_{3A}$, then $\mathcal{G} \cong \mathcal{G}V_{F(3A)}$ and $\dim \mathcal{G} = 12$.*

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