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大腰圍平面圖的強邊著色數之精確值
On the precise value of the strong chromatic-index of a planar graph with a large girth

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摘要

一個圖 G 的 k-強邊著色指的是使得距離為二以內的邊都塗不同顏色的 k-邊著色；強邊著色數 $\chi'_s(G)$ 則標明參數 k 的最小可能。此概念最初是為了解決平地上設置廣播網路的問題，由 Fouquet 與 Jolivet 提出。對於任意圖 G，參數 $\sigma(G) = \max_{xy \in E(G)}\{\deg(x) + \deg(y) - 1\}$ 是強邊著色數的一個下界；且若 G 是樹，則強邊著色數會到達此下界。另一方面，對於最大度數為 Δ 的平面圖 G，經由四色定理可以證得 $\chi'_s(G) \leq 4\Delta + 4$。更進一步，在各種腰圍與最大度數的條件下，平面圖的強邊著色數之上界分別有 $4\Delta, 3\Delta + 5, 3\Delta + 1, 3\Delta$ 和 $2\Delta - 1$ 等等優化。本篇論文說明當平面圖 G 的腰圍夠大，且 $\sigma(G) \geq \Delta(G) + 2$ 時，參數 $\sigma(G)$ 就會恰好是此圖的強邊著色數。本結果反映出大腰圍的平面圖局部上有看似樹的結構。

關鍵詞：強邊著色數、平面圖、腰圍。

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Abstract

A *strong k-edge-coloring* of a graph G is a mapping from the edge set $E(G)$ to $\{1, 2, \ldots, k\}$ such that every pair of distinct edges at distance at most two receive different colors. The *strong chromatic index* $\chi'_s(G)$ of a graph G is the minimum k for which G has a strong k-edge-coloring. The concept of strong edge-coloring was introduced by Fouquet and Jolivet to model the channel assignment in some radio networks. Denote the parameter $\sigma(G) = \max_{xy \in E(G)}\{\deg(x) + \deg(y) - 1\}$. It is easy to see that $\sigma(G) \leq \chi'_s(G)$ for any graph G, and the equality holds when G is a tree. For a planar graph G of maximum degree Δ, it was proved that $\chi'_s(G) \leq 4\Delta + 4$ by using the Four Color Theorem. The upper bound was then reduced to 4Δ, $3\Delta+5$, $3\Delta+1$, 3Δ, $2\Delta - 1$ under different conditions for Δ and the girth. In this paper, we prove that if the girth of a planar graph G is large enough and $\sigma(G) \geq \Delta(G) + 2$, then the strong chromatic index of G is precisely $\sigma(G)$. This result reflects the intuition that a planar graph with a large girth locally looks like a tree.

Keywords: Strong chromatic index, planar graph, girth.

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1 Introduction

A *strong k-edge-coloring* of a graph G is a mapping from $E(G)$ to $\{1, 2, \ldots, k\}$ such that every pair of distinct edges at distance at most two receive different colors. It induces a proper vertex coloring of $L(G)^2$, the square of the line graph of G. The *strong chromatic index* $\chi'_s(G)$ of G is the minimum k for which G has a strong k-edge-coloring. This concept was introduced by Fouquet and Jolivet [19, 20] to model the channel assignment in some radio networks. For more applications, see [4, 29, 32, 31, 24, 36].

A Vizing-type problem was asked by Erdős and Nešetřil, and further strengthened by Faudree, Schelp, Gyárfás and Tuza to give an upper bound for $\chi'_s(G)$ in terms of the maximum degree $\Delta = \Delta(G)$:

Conjecture 1 (Erdős and Nešetřil '88 [16] '89 [17], Faudree *et al.* '90 [18])**.** *If G is a graph with maximum degree Δ, then $\chi'_s(G) \leq \Delta^2 + \lfloor \frac{\Delta}{2} \rfloor^2$.*

As demonstrated in [18], there are indeed some graphs reach the given upper bounds.

By a greedy algorithm, it can be easily seen that $\chi'_s(G) \leq 2\Delta(\Delta - 1) + 1$. Molloy and Reed [28] used a probabilistic method to show that $\chi'_s(G) \leq 1.998\Delta^2$ for maximum degree Δ large enough. Recently, this upper bound was improved by Bruhn and Joos [8] to $1.93\Delta^2$.

For small maximum degrees, the cases $\Delta = 3$ and 4 were studied. Andersen [1] and Horák *et al.* [22] proved that $\chi'_s(G) \leq 10$ for $\Delta(G) \leq 3$ independently; and Cranston [13] showed that $\chi'_s(G) \leq 22$ when $\Delta(G) \leq 4$.

According to the examples in [18], the bound is tight for $\Delta = 3$, and the best we may expect for $\Delta = 4$ is 20.

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The strong chromatic index of a few families of graphs are examined, such as cycles, trees, d-dimensional cubes, chordal graphs, Kneser graphs, k-degenerate graphs, chordless graphs and C_4-free graphs, see [5, 12, 15, 18, 27, 39, 41]. As for Halin graphs, refer to [10, 25, 26, 34, 35]. For the relation to various graph products, see [37].

Now we turn to planar graphs.

Faudree *et al.* used the Four Color Theorem [2, 3] to prove that planar graphs with maximum degree Δ are strong $(4\Delta+4)$-edge-colorable [18]. By the same spirit, it can be shown that K_5-minor free graphs are strong $(4\Delta + 4)$-edge-colorable. Moreover, every planar graph G with girth at least 7 and $\Delta \geq 7$ is strong 3Δ-edge-colorable by applying a strengthened version of Vizing's Theorem on planar graphs [33, 38] and Grötzsch's theorem [21].

The following results are obtained by using a discharging method:

Theorem 2 (Hudák *et al.* '14 [23]). *If G is a planar graph with girth at least 7, then $\chi'_s(G) \leq 3\Delta(G)$.*

Theorem 3 (Bensmail *et al.* '14 [6]). *If G is a planar graph with girth at least 6, then $\chi'_s(G) \leq 3\Delta(G) + 1$.*

Theorem 4 (Bensmail *et al.* '14 [6]). *If G is a planar graph with girth at least 5 or maximum degree at least 7, then $\chi'_s(G) \leq 4\Delta(G)$.*

It is also interesting to see the asymptotic behavior of strong chromatic index when the girth is large enough.

Theorem 5 (Borodin and Ivanova '13 [7]). *If G is a planar graph with maximum degree $\Delta \geq 3$ and girth at least $40\lfloor\frac{\Delta}{2}\rfloor + 1$, then $\chi'_s(G) \leq 2\Delta - 1$.*

Theorem 6 (Chang *et al.* '14 [11]). *If G is a planar graph with maximum degree $\Delta \geq 4$ and girth at least $10\Delta + 46$, then $\chi'_s(G) \leq 2\Delta - 1$.*

Theorem 7 (Wang and Zhao '15 [40]). *If G is a planar graph with maximum degree $\Delta \geq 4$ and girth at least $10\Delta - 4$, then $\chi'_s(G) \leq 2\Delta - 1$.*

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The concept of maximum average degree is also an indicator of the sparsity of a graph. Graphs with small maximum average degrees are in relation to planar graphs with large girths, as a folklore lemma that can be proved by Euler's formula points out.

Lemma 8. *A planar graph G with girth g has maximum average degree* $\mathrm{mad}(G) < 2 + \frac{4}{g-2}$.

Many results concerning planar graphs with large girths can be extended to more general graphs with small maximum average degrees. Strong chromatic index is no exception.

Theorem 9 (Wang and Zhao '15 [40])**.** *Let G be a graph with maximum degree $\Delta \geq 4$. If the maximum average degree* $\mathrm{mad}(G) < 2 + \frac{1}{3\Delta-2}$*, the even girth is at least 6 and the odd girth is at least $2\Delta - 1$, then $\chi'_s(G) \leq 2\Delta - 1$.*

In terms of maximum degree Δ, the bound $2\Delta - 1$ is best possible. We seek for a better parameter as a refinement. Define

$$\sigma(G) := \max_{xy \in E(G)} \{\deg(x) + \deg(y) - 1\}.$$

An *antimatching* is an edge set $S \subseteq E(G)$ in which any two edges are at distance at most 2, thus any strong edge-coloring assigns distinct colors on S. Notice that each color set of a strong edge-coloring is an induced matching, and the intersection of an induced matching and an antimatching contains at most one edge. The fact suggests a dual problem to strong edge-coloring: finding a maximum antimatching of G, whose size is denoted by $\mathrm{am}(G)$. For any edge $xy \in E(G)$, the edges incident with xy form an antimatching of size $\deg(x) + \deg(y) - 1$. Together with the weak duality, this gives the inequality

$$\chi'_s(G) \geq \mathrm{am}(G) \geq \sigma(G).$$

By induction, we see that for any nontrivial tree T, $\chi'_s(T) = \sigma(T)$ attains the lower bound [18]. Based on the intuition that a planar graph with large girth locally looks like a tree, in this paper, we focus on this class of graphs. More precisely, we prove the following main theorem:

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Theorem 10. *If G is a planar graph with $\sigma = \sigma(G) \geq 5$, $\sigma \geq \Delta(G) + 2$ and girth at least $5\sigma + 16$, then $\chi'_s(G) = \sigma$.*

We also make refinement on the girth constraint and gain a stronger result in Section 4.

The condition $\sigma \geq \Delta(G)+2$ is necessary as shown in the following example. Suppose $n \geq 1$ and $d \geq 2$. Construct $G_{3n+1,d}$ from the cycle $(x_1, x_2, \ldots, x_{3n+1})$ by adding $d-2$ leaves adjacent to each x_{3i} for $1 \leq i \leq n$. Then $\sigma(G_{3n+1,d}) = d+1 < d+2 = \Delta(G_{3n+1,d}) + 2$. See Figure 1.1 for $G_{3n+1,4}$.

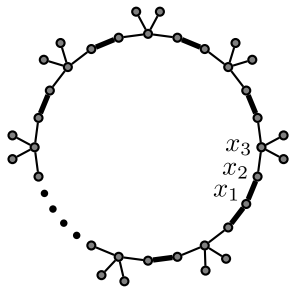

Figure 1.1: The graph $G_{3n+1,4}$.

We claim that $\sigma(G_{3n+1,d}) < \chi'_s(G_{3n+1,d})$. Suppose to the contrary that $\sigma(G_{3n+1,d}) = \chi'_s(G_{3n+1,d})$. For $1 \leq i \leq n$, the $\sigma - 1$ edges incident to x_{3i}, together with the edge $x_{3i-2}x_{3i-1}$ (or $x_{3i+1}x_{3i+2}$) use all the σ colors, implying that $x_{3i-2}x_{3i-1}$ uses the same color as $x_{3i+1}x_{3i+2}$, where $x_{3n+2} = x_1$. Therefore, $x_1x_2, x_4x_5, \ldots, x_{3n+1}x_{3n+2}$ all use the same color, contradicting that x_1x_2 is adjacent to $x_{3n+1}x_1 = x_{3n+1}x_{3n+2}$.

However, we have a corollary to remedy the situation a bit:

Corollary 11. *If G is a planar graph with $\sigma = \sigma(G) \geq 4$, $\sigma = \Delta(G) + 1$ and girth at least $5\sigma + 21$, then $\chi'_s(G) \leq \sigma + 1$.*

Proof. There must be some vertex $x \in V(G)$ of degree 2 and adjacent to another vertex of maximum degree in G. We add a pendant edge at x such that the resulting graph $\widetilde{G}$ has $\sigma(\widetilde{G}) = \sigma+1 = \Delta(G)+2 = \Delta(\widetilde{G})+2$. Now $\widetilde{G}$ satisfies the requirements of Theorem 10. Hence it is $(\sigma + 1)$-strong edge-colorable, and so is its subgraph G. □

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2 The proof of the main theorem

To prove the main theorem, we need two lemmas and a key lemma, Lemma 18, to be verified in the next section.

The first lemma can be used to prove that any tree T has strong chromatic index $\sigma(T)$ by induction.

Lemma 12. *Suppose x_1x_2 is a cut edge of a graph G, and G_i is the component of $G - x_1x_2$ containing x_i joining the edge x_1x_2 for $i = 1, 2$. If for some integer k,* $\deg(x_1) + \deg(x_2) - 1 \le k$ *and $\chi'_s(G_i) \le k$ for $i = 1, 2$, then $\chi'_s(G) \le k$.*

Proof. Choose a strong k-edge-coloring f_i of G_i for $i = 1, 2$. Let E_i be the set of edges incident with x_i in $G_i - x_1x_2$ and $S_i = f_i(E_i)$. Since $\deg(x_1) + \deg(x_2) - 1 \le k$, we may assume S_1 and S_2 are disjoint and $f_1(x_1x_2) = f_2(x_1x_2)$ is some element $c \in \{1, 2, \ldots, k\} \setminus (S_1 \cup S_2)$. Then

$$f(e) = \begin{cases} f_1(e), & \text{if } e \in E(G_1) - x_1x_2; \\ f_2(e), & \text{if } e \in E(G_2) - x_1x_2; \\ c, & \text{if } e = x_1x_2 \end{cases}$$

is a strong k-edge-coloring of G. □

The following lemma about planar graphs is also useful in the proof of the main theorem. An ℓ-thread is an induced path of $\ell + 2$ vertices all of whose internal vertices are of degree 2 in the full graph.

Lemma 13 (Nešetřil *et al.*'97 [30]). *Any planar graph G with minimum degree at least* 2 *and with girth at least $5\ell + 1$ contains an ℓ-thread.*

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Proof. Contract all the vertices of degree 2 to obtain G'. Notice that G' is a planar graph which may have multi-edges and may be disconnected. Embed $G' = (V, E)$ in the plane as P. Then Euler's Theorem says that $|V| - |E| + |F| \geq 2$, where F is the set of faces of P. If G' has girth larger than 5, we have $2|E| = \sum_{f \in F} \deg(f) \geq 6|F|$. But that G' has no vertices of degree 2 implies $2|E| = \sum_{v \in V} \deg(v) \geq 3|V|$. Combining all these produces a contradiction:

$$2 \leq |V| - |E| + |F| \leq \frac{2}{3}|E| - |E| + \frac{1}{3}|E| = 0.$$

Hence G' has a cycle of length at most 5. The corresponding cycle in G has length at least $5\ell + 1$. Thus one of these edges in G' is contracted from ℓ vertices in G, and so G has the required path. □

These two lemmas, together with Lemma 18 in the next section, lead to the following proof of the main theorem:

Proof of Theorem 10. Since the inequality $\chi'_s(G) \geq \sigma$ is trivial, it suffices to show that $\chi'_s(G) \leq \sigma$. That is, G admits a strong σ-edge-coloring φ. Suppose to the contrary that there is a counterexample G with fewest number of non-leaf vertices.

Notice that any proper subgraph of G with fewer non-leaf vertices than G admits a strong σ-edge coloring. This follows from the minimality of G, unless the proper subgraph G' does not satisfy the condition $\sigma(G') \geq \Delta(G') + 2$. However, it implies that $\sigma(G') < \Delta(G') + 2 \leq \Delta(G) + 2 \leq \sigma$. The equality $\sigma(G') = \Delta(G')$ means G' is a star, which is obviously σ-strong edge-colorable. As for the case $\sigma(G') = \Delta(G') + 1$, although Corollary 11 is derived from this theorem, it is still valid to be used since the proof only requires the graph $\widetilde{G'}$, obtained by joining a leaf to G', to be $\sigma(\widetilde{G'})$-strong edge-colorable, which is true as there are indeed fewer non-leaf vertices in $\widetilde{G'}$ than in G. So $\chi'_s(G') \leq \sigma(G') + 1 \leq \sigma$.

As a consequence, if G is not a star, then there is no non-leaf vertex x adjacent to $\deg(x) - 1$ leaves. For otherwise, there is a cut edge xy, where y is not a leaf. By applying Lemma 12 to G with the cut edge xy, we get a contradiction.

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Consider $H = G - \{x \in V(G) : \deg(x) = 1\}$, which clearly has the same girth as G since the deletion doesn't break any cycle. And we have the minimum degree $\delta(H) \geq 2$, for otherwise G has a vertex x adjacent to $\deg(x) - 1$ leaves, which is impossible as noted above. Lemma 13 claims that there is a path $x_0 x_1 \ldots x_{\ell+1}$ with $\ell = \sigma + 3$ and $\deg_H(x_i) = 2$ for $i = 1, 2, \ldots, \ell$. Now let G' be the subgraph obtained from G by deleting the leaf-neighbors of $x_1, x_2, \ldots, x_\ell$ and the vertices $x_2, x_3, \ldots, x_{\ell-1}$. This subgraph has fewer non-leaf vertices than G, so it admits a strong σ-edge-coloring φ_1. Consider the subgraph T of G induced by $x_1, x_2, \ldots, x_\ell$ and their neighbors, which is a caterpillar tree. By Lemma 18 that will be proved in the next section, T admits a strong σ-edge-coloring φ_2 such that φ_1 and φ_2 coincides on the edges $x_0 x_1$ and $x_\ell x_{\ell+1}$; furthermore, the edges incident to x_0 and x_1 all receive different colors, and so do the edges incident to x_ℓ and $x_{\ell+1}$. By gluing these two edge-colorings we construct a strong σ-edge-coloring of G. $\square$

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3 The key lemma: caterpillar with edge pre-coloring

The purpose of this section is to prove the key lemma, Lemma 18, in this thesis.

All the graphs in this section are caterpillar trees. Let $d_i \geq 2$ for $i = 1, 2, \ldots, \ell$. By $T = \text{Cat}(d_1, d_2, \ldots, d_\ell)$ we mean a caterpillar tree with spine $x_0, x_1, \ldots, x_{\ell+1}$, whose degrees are $d_0, d_1, \ldots, d_{\ell+1}$, where $d_0 = d_{\ell+1} = 1$. Call ℓ the length of T and let E_i be the edges incident with x_i. See Figure 3.1 for Cat(5,3,2,4,5).

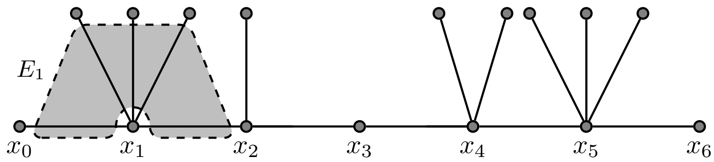

Figure 3.1: The caterpillar tree Cat(5,3,2,4,5).

For color sets C_1 and C_2, denote $C_1 - C_2 := C_1 \backslash C_2$ the difference of the two sets. If $C_2 = \{\alpha\}$ contains only one element, we also denote it by $C_1 - \alpha$.

Collect all the tuples $(C; \alpha_0, C_1, C_\ell, \alpha_\ell)$ as $\mathcal{P}_\kappa(T)$, where the color sets $C_1, C_\ell \subseteq C$ with $|C_1| = d_1, |C_\ell| = d_\ell$, $|C| = \kappa$, and $\alpha_0 \in C_1, \alpha_\ell \in C_\ell$. Fix $\kappa \in \mathbb{N}$. For any $P = (C; \alpha_0, C_1, C_\ell, \alpha_\ell) \in \mathcal{P}_\kappa(T)$, the set of all strong edge-colorings φ using the colors in C and satisfying the following criterions is denoted by $\mathcal{C}_T(P)$:

$$\varphi(E_1) = C_1, \quad \varphi(E_\ell) = C_\ell, \quad \varphi(x_0x_1) = \alpha_0 \quad \text{and} \quad \varphi(x_\ell x_{\ell+1}) = \alpha_\ell.$$

If $\mathcal{C}_T(P)$ is nonempty for any $P \in \mathcal{P}_\kappa(T)$ with $\kappa \geq \sigma(T)$, then T is said to be *strong κ-edge-colorable with two-sided pre-coloring.*

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Lemma 14. *If $T = \text{Cat}(d_1, d_2, \ldots, d_\ell)$ is strong κ-edge-colorable with two-sided pre-coloring, then T is strong κ'-edge-colorable with two-sided pre-coloring for any $\kappa' \geq \kappa$.*

Proof. For any $P' = (C'; \alpha'_0, C'_1, C'_\ell, \alpha'_\ell) \in \mathcal{P}_{\kappa'}(T)$, we have to find a strong edge-coloring in $\mathcal{C}_T(P')$.

Case $|C'_1 \cup C'_\ell| \leq \kappa$: Choose a κ-set C so that $C'_1 \cup C'_\ell \subseteq C \subseteq C'$. By assumption, there is a strong edge-coloring in $\mathcal{C}_T(C; \alpha'_0, C'_1, C'_\ell, \alpha'_\ell) \subseteq \mathcal{C}_T(P')$.

Case $|C'_1 \cup C'_\ell| > \kappa$: Choose a κ-set C so that $C'_1 \cup \{\alpha'_\ell\} \subseteq C \subseteq C'_1 \cup C'_\ell$, and a d_ℓ-set C_ℓ so that $C'_\ell \cap C \subseteq C_\ell \subseteq C$. By assumption, there is a strong edge-coloring φ in $\mathcal{C}_T(C; \alpha'_0, C'_1, C_\ell, \alpha'_\ell)$. Let the edges in E_ℓ with color $C_\ell - C'_\ell$ be E'_ℓ. Notice $C'_\ell - C_\ell$ and C are disjoint, so the colors in $C'_\ell - C_\ell$ are not appeared in φ. Hence we can change the colors of E'_ℓ to $C'_\ell - C_\ell$ and obtain a strong edge-coloring in $\mathcal{C}_T(P')$. □

We now derive a series of properties regarding the strong edge-pre-colorability with two-sided pre-coloring of a caterpillar tree and its certain subtrees.

Lemma 15. *Suppose a caterpillar tree $\widetilde{T}$ contains T as a subgraph, and both have the same length. If $\widetilde{T}$ is strong κ-edge-colorable with two-sided pre-coloring, then so is T.*

Proof. Suppose $(C; \alpha_0, C_1, C_\ell, \alpha_\ell) \in \mathcal{P}_\kappa(T)$. We find $(C; \alpha_0, C'_1, C'_\ell, \alpha_\ell) \in \mathcal{P}_\kappa(\widetilde{T})$ such that $C'_1 \supseteq C_1$ and $C'_\ell \supseteq C_\ell$. The lemma follows that any $\varphi' \in \mathcal{C}_{\widetilde{T}}(C; \alpha_0, C'_1, C'_\ell, \alpha_\ell)$ has a restriction φ on T so that φ is a strong edge-coloring in $\mathcal{C}_T(C; \alpha_0, C_1, C_\ell, \alpha_\ell)$. □

For $T = \text{Cat}(d_1, d_2, \ldots, d_\ell)$, let T_{-1} be the subtree $\text{Cat}(d_1, d_2, \ldots, d_{\ell-1})$.

Lemma 16. *For a caterpillar tree $T = \text{Cat}(d_1, d_2, \ldots, d_\ell)$, if T_{-1} is strong κ-edge-colorable with two-sided pre-coloring, where $\kappa \geq \sigma(T)$, then so is T.*

Proof. For any $P = (C; \alpha_0, C_1, C_\ell, \alpha_\ell) \in \mathcal{P}_\kappa(T)$, pick $\alpha_{\ell-1} \in C_\ell - \alpha_\ell$ and $C_{\ell-1}$ a $d_{\ell-1}$-subset of C with $C_{\ell-1} \cap C_\ell = \{\alpha_{\ell-1}\}$. Notice that $C_{\ell-1}$ can be chosen since $d_{\ell-1} + d_\ell - 1 \leq \sigma(T) \leq \kappa$. By the assumption, T_{-1} admits a strong κ-edge-coloring $\varphi \in \mathcal{C}_{T_{-1}}(C; \alpha_0, C_1, C_{\ell-1}, \alpha_{\ell-1})$. Coloring the remaining edges with $C_\ell - \alpha_{\ell-1}$ so that $x_\ell x_{\ell+1}$ has color α_ℓ results in a strong κ-edge-coloring in $\mathcal{C}_T(P)$. □

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Hereafter, if necessary we reverse the order to view $T = \text{Cat}(d_\ell, d_{\ell-1}, \dots, d_1)$ so that we can always assume $\sigma(T_{-1}) = \sigma(T)$. Hence the requirement $\kappa \geq \sigma(T)$ in Lemma 16 automatically holds.

For a caterpillar tree T, we define T' and I_T as follows. Call a vertex x_i *σ-large* if $d_i \geq d^* := \lceil \frac{\sigma+1}{2} \rceil$. The value d^* is critical in the sense that

1. If $d_i + d_j \leq \sigma + 1$, then either d_i or d_j must be at most d^*.

2. If $d_i + d_j \geq \sigma + 1$, then either d_i or d_j must be at least d^*.

Let $S = \{x_i : i \in I_T\}$ be the set of all σ-large vertices, except that if there exist $i < j$ with $d_{i-1} < d^*$, $d_i = d_{i+1} = \ldots = d_j = d^*$ and $d_{j+1} < d^*$, we only take $x_i, x_{i+2}, x_{i+4}, \ldots$ till x_j or x_{j-1}, depending on the parity. Then S is a nonempty independent set. Consider a new degree sequence $d'_1, d'_2, \ldots, d'_\ell$ where

$$d'_i = \begin{cases} d_i - 1, & \text{if } i \in I_T; \\ d_i, & \text{if } i \notin I_T. \end{cases}$$

Then $T' = \text{Cat}(d'_1, d'_2, \ldots, d'_\ell)$ is a caterpillar tree isomorphic to a subgraph of T, with $\sigma(T') = \sigma(T) - 1$ due to the criticalness of d^* and the choice method of S.

It is straightforward to see that $(T')_{-1} = (T_{-1})' = \text{Cat}(d'_1, d'_2, \ldots, d'_{\ell-1})$ by the choice method of S, and we denote it by T'_{-1} for short.

Lemma 17. *For $T = \text{Cat}(d_1, d_2, \ldots, d_\ell)$, suppose $\sigma = \sigma(T) = \sigma(T_{-1}) \geq 6$ and T'_{-1} is strong $(\sigma - 1)$-edge-colorable with two-sided pre-coloring, then T is strong σ-edge-colorable with two-sided pre-coloring.*

Proof. For any $P = (C; \alpha_0, C_1, C_\ell, \alpha_\ell) \in \mathcal{P}_\sigma(T)$, we must show that $\mathcal{C}_T(P)$ is nonempty.

Let $I = I_T$. Our strategy is to search for a color β such that

$$\beta \in C_1 \text{ if and only if } 1 \in I; \text{ and } \beta \in C_\ell \text{ if and only if } \ell \in I.$$

Suppose such a color β exists and $\beta \neq \alpha_\ell$. By Lemma 16, T' admits a strong $(\sigma - 1)$-edge coloring in $\mathcal{C}_{T'}(C - \beta; \alpha_0, C_1 - \beta, C_\ell - \beta, \alpha_\ell)$. Coloring the remaining edges with β then

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yields the required strong κ-edge-coloring in $\mathcal{C}_T(P)$. Notice that S being an independent set guarantees that the edges with color β form an induced matching. If it happens that β coincides with α_ℓ, then we seek instead for strong-edge coloring in $\mathcal{C}_{T'}(C-\beta;\alpha_0, C_1-\beta, C_\ell-\beta, \alpha'_\ell)$ for arbitrary $\alpha'_\ell \in C_\ell - \alpha_\ell$. We make use of the symmetry of pendant edges incident with x_ℓ and still achieve the goal.

Sometimes there is no suitable β. We alternatively consider T_{-1}. By finding appropriate $d_{\ell-1}$-subset $C_{\ell-1} \subseteq C$ and $\alpha_{\ell-1}$ with $C_{\ell-1} \cap C_\ell = \{\alpha_{\ell-1}\}$, there will be a β such that

$$\beta \in C_1 \text{ if and only if } 1 \in I; \text{ and } \beta \in C_{\ell-1} \text{ if and only if } \ell - 1 \in I.$$

Similarly, there is a strong edge-coloring in $\mathcal{C}_{T_{-1}}(C;\alpha_0, C_1, C_{\ell-1}, \alpha_{\ell-1})$, as T'_{-1} is strong $(\sigma-1)$-edge-colorable with two-sided pre-coloring. Color the remaining edges with $C_\ell - \alpha_{\ell-1}$ so that $x_\ell x_{\ell+1}$ has color α_ℓ, we gain a strong σ-edge-coloring in $\mathcal{C}_T(P)$.

We now prove the existence of β according to the following four cases.

Case 1. $1, \ell \in I$. In this case, $C_1 \cap C_\ell$ is nonempty since

$$|C_1 \cap C_\ell| = |C_1| + |C_\ell| - |C_1 \cup C_\ell| \geq 2d^* - \sigma > 0.$$

Pick β to be any color in the intersection.

Case 2. $1 \in I$ but $\ell \notin I$. If $C_1 - C_\ell$ is nonempty, then pick β to be any color in the difference. Otherwise, $1 \in I$ and $\ell \notin I$ imply $d_1 \geq d^* \geq d_\ell$. On the other hand, $C_1 - C_\ell = \emptyset$ implies $d_1 \leq d_\ell$. Thus the situation that $C_1 - C_\ell$ is empty occurs only when $d_1 = d_\ell = d^*$ and $C_1 = C_\ell$. We consider the subtree T_{-1}. Choose $\alpha_{\ell-1}$ to be any color in $C_\ell - \alpha_\ell$. Let $C_{\ell-1}$ be $\alpha_{\ell-1}$ together with any $(d_{\ell-1}-1)$-subset in $C - C_\ell$.

Since $d_\ell = d^*$ but $\ell \notin I$, it is the case that $\ell - 1 \in I$ and $d_{\ell-1} = d^*$. Pick $\beta = \alpha_{\ell-1}$.

Case 3. $\ell \in I$ but $1 \notin I$. If $C_\ell - C_1$ is nonempty, then let β be any color in the difference. Otherwise, $d_1 = d_\ell = d^*$ and $C_1 = C_\ell$. But $d_1 = d^*$ implies $1 \in I$, a contradiction.

Case 4. $1, \ell \notin I$. If $C - (C_1 \cup C_\ell)$ is nonempty, then pick β to be any color in the difference set. Now, suppose $C = C_1 \cup C_\ell$. We consider the subtree T_{-1}.

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First estimate the size

$$|C_\ell - C_1| = |C_\ell \cup C_1| - |C_1| \geq \sigma - d^* \geq d^* - 2 \geq 2,$$

where $d^* \geq 4$ since $\sigma \geq 6$. Pick $\alpha_{\ell-1}$ to be any color in $C_\ell - C_1 - \alpha_\ell$. Let $C_{\ell-1}$ be a color set such that $|C_{\ell-1}| = d_{\ell-1}$ and $C_{\ell-1} \cap C_\ell = \{\alpha_{\ell-1}\}$.

When $\ell - 1 \in I$, pick $\beta = \alpha_{\ell-1}$. Otherwise, let β be chosen from $C_\ell - C_1 - \alpha_{\ell-1}$. $\square$

Now we are ready to prove the key lemma.

Lemma 18. *Suppose $T = \text{Cat}(d_1, d_2, \ldots, d_\ell)$ is a nice caterpillar tree, i.e. it satisfies*

$$\sigma = \sigma(T) \geq 5, \quad \ell \geq \sigma + 3 \quad \text{and} \quad \sigma \geq \Delta(T) + 2.$$

For any $\kappa \geq \sigma(T)$, any color sets $C_1, C_\ell \subseteq C$ with $|C| = \kappa$, $|C_1| = d_1$, $|C_\ell| = d_\ell$, and any two colors $\alpha_0 \in C_1$, $\alpha_\ell \in C_\ell$, there is a strong σ-edge coloring φ using the colors in C such that $\varphi(E_1) = C_1$, $\varphi(E_\ell) = C_\ell$ and $\varphi(x_0x_1) = \alpha_0$, $\varphi(x_\ell x_{\ell+1}) = \alpha_\ell$. That is, T is strong κ-edge-colorable with two-sided pre-coloring for any $\kappa \geq \sigma$.

Proof. We prove the lemma by induction on $\sigma = \sigma(T)$. Recall that we always assume the condition $\sigma(T_{-1}) = \sigma(T)$ holds. By Lemmas 14 and 16, it suffices to consider the case $\kappa = \sigma$ and $\ell = \sigma + 3$.

If T is nice and $\sigma \geq 6$, then T'_{-1} is also a nice caterpillar tree: The first two conditions remain since $\sigma(T'_{-1}) = \sigma(T') = \sigma(T) - 1$. The third one $\sigma(T'_{-1}) \geq \Delta(T'_{-1}) + 2$ fails only when $\sigma(T) = \sigma(T'_{-1}) + 1 \leq \Delta(T'_{-1}) + 2 \leq \Delta(T) + 2$ and so $\Delta(T') = \Delta(T)$. Since $\Delta(T) \geq d^*$, in this case, $\Delta(T) = d^* \geq 4$ and there is at least a pair of consecutive vertices with $d_i = d_{i+1} = d^*$. Then $\sigma(T'_{-1}) = \sigma(T) - 1 = 2\Delta(T) - 2 \geq \Delta(T) + 2 \geq \Delta(T'_{-1}) + 2$.

By Lemma 17, we only have to discuss the base cases $\sigma = 5$ and $\ell = 8$. We may assume all degrees $d_i = 3$ since $\sigma \geq \Delta + 2$. Also assume $C_1 = \{1, 2, 3\}$ and $\alpha_0 = 1$. Depending on $C_1 \cap C_8$ and whether $\alpha_8 = \alpha_0$ or not, by symmetry we color T according to φ shown in Table 3.1, where $\alpha_i = \varphi(x_i x_{i+1})$ and $\widehat{C}_i = \varphi(C_i) - \varphi(x_{i-1}x_i) - \varphi(x_i x_{i+1})$. Or we can solve this case by the argument in [7] or the odd graph method in [11, 40]. $\square$

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α_0	$\widehat{C}_1$	α_1	$\widehat{C}_2$	α_2	$\widehat{C}_3$	α_3	$\widehat{C}_4$	α_4	$\widehat{C}_5$	α_5	$\widehat{C}_6$	α_6	$\widehat{C}_7$	α_7	$\widehat{C}_8$	α_8
1	$\{3\}$	2	$\{4\}$	5	$\{1\}$	3	$\{4\}$	2	$\{5\}$	1	$\{3\}$	4	$\{5\}$	2	$\{1\}$	3
1	$\{3\}$	2	$\{4\}$	5	$\{1\}$	3	$\{4\}$	2	$\{5\}$	1	$\{3\}$	4	$\{5\}$	2	$\{3\}$	1
1	$\{2\}$	3	$\{5\}$	4	$\{1\}$	2	$\{5\}$	3	$\{1\}$	4	$\{2\}$	5	$\{1\}$	3	$\{4\}$	2
1	$\{2\}$	3	$\{5\}$	4	$\{1\}$	2	$\{5\}$	3	$\{1\}$	4	$\{2\}$	5	$\{1\}$	3	$\{2\}$	4
1	$\{3\}$	2	$\{4\}$	5	$\{1\}$	3	$\{4\}$	2	$\{5\}$	1	$\{4\}$	3	$\{5\}$	2	$\{4\}$	1
1	$\{3\}$	2	$\{4\}$	5	$\{1\}$	3	$\{4\}$	2	$\{5\}$	1	$\{4\}$	3	$\{5\}$	2	$\{1\}$	4
1	$\{3\}$	2	$\{4\}$	5	$\{1\}$	3	$\{2\}$	4	$\{5\}$	1	$\{2\}$	3	$\{5\}$	4	$\{1\}$	2
1	$\{3\}$	2	$\{4\}$	5	$\{1\}$	3	$\{4\}$	2	$\{5\}$	1	$\{4\}$	3	$\{2\}$	5	$\{4\}$	1
1	$\{3\}$	2	$\{4\}$	5	$\{1\}$	3	$\{4\}$	2	$\{5\}$	1	$\{4\}$	3	$\{2\}$	5	$\{1\}$	4
1	$\{3\}$	2	$\{4\}$	5	$\{1\}$	3	$\{2\}$	4	$\{1\}$	5	$\{2\}$	3	$\{1\}$	4	$\{5\}$	2
1	$\{3\}$	2	$\{4\}$	5	$\{1\}$	3	$\{2\}$	4	$\{1\}$	5	$\{2\}$	3	$\{1\}$	4	$\{2\}$	5

Table 3.1: The 5-strong edge-colorings of T for $\sigma = 5$ with $\ell = 8$.

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4 Refinement of the key lemma and its consequences

We now discuss the optimality of Lemma 18. If we take more care about the base cases, there would be a refinement:

Lemma 19. *Suppose T is a caterpillar tree of length ℓ satisfying*

$$\sigma = \sigma(T) \geq 5, \quad \ell \geq \ell_\sigma \quad \text{and} \quad \sigma \geq \Delta(T) + 2,$$

where

$$\ell_\sigma = \begin{cases} 8, & \textit{if } \sigma = 5; \\ 7, & \textit{if } \sigma = 6; \\ \sigma, & \textit{if } \sigma \geq 7. \end{cases}$$

Then T is strong κ-edge-colorable with two-sided pre-coloring for any $\kappa \geq \sigma$.

Proof. Similar to Lemma 18, we only need to consider the base cases.

For $\sigma = 6$, we first consider the situation $\ell = 6$. By Lemma 15 and the symmetry, it suffices to discuss the caterpillar trees Cat$(4, 3, 4, 3, 4, 3)$, Cat$(4, 3, 4, 3, 3, 4)$, and Cat$(3, 4, 3, 3, 4, 3)$. We enumerate all the cases in Table 4.1 and Table 4.2 to show that the first two are strong 6-edge-colorable with two-sided pre-coloring.

If the caterpillar tree T considered with $\sigma = 6$ and $\ell = 7$ has $T_{-1} =$ Cat$(3, 4, 3, 3, 4, 3)$, then T is a subtree of Cat$(3, 4, 3, 3, 4, 3, 4)$. We can assume $T =$ Cat$(3, 4, 3, 3, 4, 3, 4)$ by Lemma 15. Reverse the direction to see T as Cat$(4, 3, 4, 3, 3, 4, 3)$. Then the subtree $T_{-1} =$ Cat$(4, 3, 4, 3, 3, 4)$, which is strong 6-edge-colorable with two-sided pre-coloring.

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α_0	$\widehat{C}_1$	α_1	$\widehat{C}_2$	α_2	$\widehat{C}_3$	α_3	$\widehat{C}_4$	α_4	$\widehat{C}_5$	α_5	$\widehat{C}_6$	α_6
1	$\{3,4\}$	2	$\{6\}$	5	$\{3,4\}$	1	$\{2\}$	6	$\{4,5\}$	3	$\{2\}$	1
1	$\{2,4\}$	3	$\{5\}$	6	$\{1,4\}$	2	$\{3\}$	5	$\{4,5\}$	1	$\{3\}$	2
1	$\{3,4\}$	2	$\{6\}$	5	$\{3,4\}$	1	$\{2\}$	6	$\{3,4\}$	5	$\{2\}$	1
1	$\{2,4\}$	3	$\{5\}$	6	$\{1,4\}$	2	$\{5\}$	3	$\{4,6\}$	1	$\{5\}$	2
1	$\{2,4\}$	3	$\{5\}$	6	$\{2,4\}$	1	$\{5\}$	3	$\{4,6\}$	2	$\{1\}$	5
1	$\{2,4\}$	3	$\{5\}$	6	$\{2,4\}$	1	$\{5\}$	3	$\{2,4\}$	6	$\{5\}$	1
1	$\{2,3\}$	4	$\{5\}$	6	$\{2,3\}$	1	$\{5\}$	4	$\{2,3\}$	6	$\{1\}$	5
1	$\{3,4\}$	2	$\{6\}$	5	$\{1,4\}$	3	$\{2\}$	6	$\{1,5\}$	4	$\{3\}$	2
1	$\{2,3\}$	4	$\{5\}$	6	$\{1,3\}$	2	$\{5\}$	4	$\{1,6\}$	3	$\{5\}$	2
1	$\{2,3\}$	4	$\{5\}$	6	$\{1,3\}$	2	$\{5\}$	4	$\{1,6\}$	3	$\{2\}$	5
1	$\{2,3\}$	4	$\{5\}$	6	$\{1,3\}$	2	$\{5\}$	4	$\{1,3\}$	6	$\{5\}$	2
1	$\{2,4\}$	3	$\{5\}$	6	$\{1,4\}$	2	$\{5\}$	3	$\{1,4\}$	6	$\{2\}$	5

Table 4.1: The 6-strong edge-colorings for $T = \text{Cat}(4, 3, 4, 3, 4, 3)$.

α_0	$\widehat{C}_1$	α_1	$\widehat{C}_2$	α_2	$\widehat{C}_3$	α_3	$\widehat{C}_4$	α_4	$\widehat{C}_5$	α_5	$\widehat{C}_6$	α_6
1	$\{2,4\}$	3	$\{5\}$	6	$\{2,4\}$	1	$\{3\}$	5	$\{6\}$	4	$\{2,3\}$	1
1	$\{3,4\}$	2	$\{5\}$	6	$\{1,4\}$	3	$\{2\}$	5	$\{6\}$	1	$\{3,4\}$	2
1	$\{3,4\}$	2	$\{5\}$	6	$\{1,3\}$	4	$\{5\}$	2	$\{6\}$	3	$\{4,5\}$	1
1	$\{2,4\}$	3	$\{6\}$	5	$\{1,2\}$	4	$\{3\}$	6	$\{2\}$	1	$\{4,5\}$	3
1	$\{2,4\}$	3	$\{6\}$	5	$\{1,2\}$	4	$\{3\}$	6	$\{2\}$	1	$\{3,4\}$	5
1	$\{3,4\}$	2	$\{5\}$	6	$\{3,4\}$	1	$\{5\}$	2	$\{3\}$	6	$\{4,5\}$	1
1	$\{3,4\}$	2	$\{6\}$	5	$\{3,4\}$	1	$\{6\}$	2	$\{3\}$	5	$\{1,6\}$	4
1	$\{3,4\}$	2	$\{6\}$	5	$\{1,3\}$	4	$\{6\}$	2	$\{3\}$	1	$\{4,6\}$	5
1	$\{3,4\}$	2	$\{6\}$	5	$\{1,4\}$	3	$\{2\}$	6	$\{1\}$	4	$\{3,5\}$	2
1	$\{2,4\}$	3	$\{6\}$	5	$\{1,2\}$	4	$\{3\}$	6	$\{1\}$	2	$\{3,4\}$	5
1	$\{3,4\}$	2	$\{5\}$	6	$\{1,4\}$	3	$\{5\}$	2	$\{1\}$	6	$\{4,5\}$	3
1	$\{3,4\}$	2	$\{6\}$	5	$\{1,3\}$	4	$\{6\}$	2	$\{1\}$	3	$\{4,6\}$	5

Table 4.2: The 6-strong edge-colorings for $T = \text{Cat}(4, 3, 4, 3, 3, 4)$.

Hence all the caterpillar trees with $\sigma = 6$ and $\ell = 7$ are strong 6-edge-colorable with two-sided pre-coloring, as the other possibilities of T_{-1} can be dealt with by Lemma 16 directly.

For $\sigma = 7$ and $\ell = 7$. It suffices to consider the caterpillar trees in Table 4.3.

All the trees T considered except Cat$(3, 5, 3, 4, 4, 4, 4)$ and Cat$(3, 5, 3, 4, 4, 3, 5)$ have T'_{-1} being strong 6-edge-colorable with two-sided pre-coloring, so these T are strong 7-edge-colorable with two-sided pre-coloring by Lemma 17.

If we see the caterpillar tree Cat$(3, 5, 3, 4, 4, 4, 4)$ as $T = \text{Cat}(4, 4, 4, 4, 3, 5, 3)$, then $T'_{-1} = \text{Cat}(3, 4, 3, 4, 3, 4)$ is strong 6-edge-colorable with two-sided pre-coloring. Similarly, regard the caterpillar tree Cat$(3, 5, 3, 4, 4, 3, 5)$ as $T = \text{Cat}(5, 3, 4, 4, 3, 5, 3)$, then $T'_{-1} = \text{Cat}(4, 3, 3, 4, 3, 4)$ is strong 6-edge-colorable with two-sided pre-coloring. So these

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T	T'_{-1}
$\text{Cat}(3,5,3,5,3,5,3)$	$\text{Cat}(3,4,3,4,3,4)$
$\text{Cat}(5,3,5,3,3,5,3)$	$\text{Cat}(4,3,4,3,3,4)$
$\text{Cat}(5,3,3,5,3,5,3)$	$\text{Cat}(4,3,3,4,3,4)$
$\text{Cat}(5,3,5,3,5,3,5)$	$\text{Cat}(4,3,4,3,4,3)$
$\text{Cat}(5,3,3,5,3,3,5)$	$\text{Cat}(4,3,3,4,3,3)$
$\text{Cat}(3,5,3,5,3,4,4)$	$\text{Cat}(3,4,3,4,3,3)$
$\text{Cat}(5,3,5,3,4,4,4)$	$\text{Cat}(4,3,4,3,3,4)$
$\text{Cat}(3,5,3,4,4,4,4)$	$\text{Cat}(3,4,3,3,4,3)$
$\text{Cat}(5,3,4,4,4,4,4)$	$\text{Cat}(4,3,3,4,3,4)$
$\text{Cat}(4,4,4,4,4,4,4)$	$\text{Cat}(3,4,3,4,3,4)$
$\text{Cat}(3,5,3,4,4,3,5)$	$\text{Cat}(3,4,3,3,4,3)$
$\text{Cat}(5,3,4,4,4,3,5)$	$\text{Cat}(4,3,3,4,3,3)$
$\text{Cat}(4,4,3,5,3,4,4)$	$\text{Cat}(3,4,3,4,3,3)$
$\text{Cat}(4,4,3,5,3,3,5)$	$\text{Cat}(3,4,3,4,3,3)$

Table 4.3: The caterpillar trees to be considered for $\sigma = 7$ and $\ell = 7$.

two trees are also strong 7-edge-colorable with two-sided pre-coloring by Lemma 17, and hence all the caterpillar trees considered with $\sigma = 7$ and $\ell = 7$ are strong 7-edge-colorable with two-sided pre-coloring. □

The ℓ_σ here cannot be reduced: For $\sigma \geq 7$, consider $\ell = \sigma - 1$ and the caterpillar tree $T = \text{Cat}(d_1, d_2, \ldots, d_\ell)$, where $d_1, d_3 \cdots = \lfloor \frac{\sigma+1}{2} \rfloor$ and $d_2, d_4 \cdots = \lceil \frac{\sigma+1}{2} \rceil$.

If $\sigma = 2d - 1$ is an odd integer, let $P = ([1, \sigma]; 1, [1, d], [1, d], 1) \in \mathcal{P}_\sigma(T)$. Suppose there is some $\varphi \in \mathcal{C}_T(P)$. Let $C_i = \varphi(E_i)$. Then $|C_{i+2} - C_i| = 1$ for $i = 1, 2, \ldots, \ell - 2$. So

$$|C_\ell - C_2| \leq |C_\ell - C_{\ell-2}| + |C_{\ell-2} - C_{\ell-4}| + \cdots + |C_4 - C_2| \leq d - 2.$$

However, $C_1 = C_\ell$ implies $|C_\ell - C_2| = d - 1$, a contradiction.

If $\sigma = 2d - 2$ is an even integer, let $P = ([1, \sigma]; 1, [1, d-1], [d, 2d-2], d) \in \mathcal{P}_\sigma(T)$. Suppose there is some $\varphi \in \mathcal{C}_T(P)$. Let $C_i = \varphi(E_i)$. Again $|C_{i+2} - C_i| = 1$ for $i = 1, 2, \ldots, \ell - 2$. Similarly, $d - 1 = |C_\ell - C_1| \leq d - 2$, a contradiction.

For $\sigma = 6$, let $T = \text{Cat}(3, 4, 3, 3, 4, 3)$ and $P = ([1, 6]; 1, \{1, 2, 3\}, \{4, 5, 6\}, 6) \in \mathcal{P}_\sigma(T)$. Suppose there is some $\varphi \in \mathcal{C}_T(P)$. Let $C_i = \varphi(E_i)$. Then $\varphi(x_3 x_4) \in \{1, 2, 3\}$ since $C_1 \cup C_2 = C_2 \cup C_3 = [1, 6]$. Similarly, $\varphi(x_3 x_4) \in \{4, 5, 6\}$ since $C_4 \cup C_5 = C_5 \cup C_6 = [1, 6]$. A contradiction follows.

Exploiting Lemma 19, the main Theorem 10 can be strengthened to:

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Theorem 20. *If G is a planar graph with $\sigma = \sigma(G) \geq 5$, $\sigma \geq \Delta(G) + 2$ and girth at least g_σ, where*

$$g_\sigma = \begin{cases} 41, & \text{if } \sigma = 5; \\ 36, & \text{if } \sigma = 6; \\ 5\sigma + 1, & \text{if } \sigma \geq 7, \end{cases}$$

then $\chi'_s(G) = \sigma$.

If we take off the condition $\sigma \geq \Delta + 2$ in Theorem 20, a weaker result can be obtained by using the following corollary of Lemma 19 in the proof of the main Theorem 10.

Corollary 21. *Suppose T is a caterpillar tree of length ℓ satisfying*

$$\sigma = \sigma(T) \geq 4 \quad \text{and} \quad \ell \geq \ell_{\sigma+1},$$

where

$$\ell_{\sigma+1} = \begin{cases} 8, & \text{if } \sigma + 1 = 5; \\ 7, & \text{if } \sigma + 1 = 6; \\ \sigma + 1, & \text{if } \sigma + 1 \geq 8. \end{cases}$$

Then T is strong κ-edge-colorable with two-sided pre-coloring for any $\kappa \geq \sigma + 1$.

Proof. Add pendant edges at some vertices of T with degree $\delta(T)$ such that the resulting graph $\widetilde{T}$ has $\sigma(\widetilde{T}) = \sigma(T) + 1$ and $\sigma(\widetilde{T}) \geq \Delta(\widetilde{T}) + 2$. So $\widetilde{T}$ satisfies the requirements of Lemma 19, and hence it is strong κ-edge-colorable with two-sided pre-coloring for any $\kappa \geq \sigma(\widetilde{T}) = \sigma(T) + 1$. The corollary then follows from Lemma 15. □

Theorem 22. *If G is a planar graph with $\sigma = \sigma(G) \geq 4$ and girth at least $g_{\sigma+1}$, where*

$$g_{\sigma+1} = \begin{cases} 41, & \text{if } \sigma + 1 = 5; \\ 36, & \text{if } \sigma + 1 = 6, 7; \\ 5\sigma + 6, & \text{if } \sigma + 1 \geq 8, \end{cases}$$

then $\sigma \leq \chi'_s(G) \leq \sigma + 1$.

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5 Consequences concerning the maximum average degree

The following lemma is a direct consequence of Proposition 2.2 in [14].

Lemma 23 (Cranston and West '13 [14])**.** *Suppose the connected graph G is not a cycle. If G has minimum degree at least 2 and average degree $\frac{2|E|}{|V|} < 2 + \frac{2}{3\ell-1}$, then G contains an ℓ-thread.*

A C_n-jellyfish is a graph by adding pendant edges at the vertices of C_n. In [9], it is shown that

Proposition 24 (Chang *et al.*'15 [9])**.** *If G is a C_n-jellyfish of m edges with $\sigma(G) \geq 4$, then $\chi'_s(G) =$*

$$\begin{cases} m, & \textit{if } n = 3; \\ \sigma(G)+1, & \textit{if } n = 4; \\ \lceil \frac{m}{\lfloor n/2 \rfloor} \rceil, & \textit{otherwise, if } n \textit{ is odd with all } \deg(v_i) = d \textit{ but } (n,d) \neq (7,3), \\ & \qquad \textit{or with } \lceil \frac{m}{\lfloor n/2 \rfloor} \rceil \geq \sigma(G)+1; \\ \sigma(G)+1, & \textit{otherwise, if } (n,d) = (7,3) \textit{ with all } \deg(v_i) = d, \\ & \qquad \textit{or } n \not\equiv 0 \pmod 3 \textit{ such that up to rotation } \deg(v_i) = \sigma(G)-1 \\ & \qquad\quad \textit{for } i \equiv 1 \pmod 3 \textit{ with } 1 \leq i \leq 3\lfloor \frac{n}{3} \rfloor - 2, \\ & \qquad \textit{or } (n, \sigma(G)) = (10,4) \textit{ with } \deg(v_i) = 3 \\ & \qquad\quad \textit{for all odd or all even } i; \\ \sigma(G), & \textit{otherwise}. \end{cases}$$

Adopting these results leads to a strengthening of Theorem 9.

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Theorem 25. *If G is a graph with $\sigma = \sigma(G) \geq 5$, $\sigma \geq \Delta(G) + 2$, odd girth at least g'_σ, even girth at least 6, and* $\text{mad}(G) < 2 + \frac{2}{3\ell_\sigma - 1}$, *where*

$$g'_\sigma = \begin{cases} 9, & \text{if } \sigma = 5; \\ \sigma, & \text{if } \sigma > 5, \end{cases} \quad \text{and} \quad \ell_\sigma = \begin{cases} 8, & \text{if } \sigma = 5; \\ 7, & \text{if } \sigma = 6; \\ \sigma, & \text{if } \sigma \geq 7, \end{cases}$$

then $\chi'_s(G) = \sigma$.

Proof. In the proof of Theorem 20, alternatively use Lemma 23 to find an ℓ_σ-thread in H. It should be noticed the girth constraints exist merely to address the problem that H may be a cycle. In this case, by Proposition 24, G still has strong chromatic index σ.

Indeed, suppose $H = C_n$ and G is a C_n-jellyfish. The case n is even is trivial. If $\sigma \geq \sigma(H) \geq 5$, n is odd and $n \geq g'_\sigma \geq \sigma$, then

$$\left\lceil \frac{|E(G)|}{\lfloor \frac{n}{2} \rfloor} \right\rceil \leq \left\lceil \frac{\frac{n-1}{2}(\sigma - 1) + \frac{\sigma+1}{2} - 1}{\frac{n-1}{2}} \right\rceil \leq \sigma.$$

Hence $\chi'_s(G) = \sigma$. □

Similarly, Theorem 22 can be modified correspondingly.

Theorem 26. *If G is a graph with $\sigma = \sigma(G) \geq 4$, odd girth at least $\frac{\sigma+1}{2}$, and* $\text{mad}(G) < 2 + \frac{2}{3\ell_{\sigma+1} - 1}$, *where*

$$\ell_{\sigma+1} = \begin{cases} 8, & \text{if } \sigma + 1 = 5; \\ 7, & \text{if } \sigma + 1 = 6; \\ \sigma + 1, & \text{if } \sigma + 1 \geq 7, \end{cases}$$

then $\sigma \leq \chi'_s(G) \leq \sigma + 1$.

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